J Stat Phys (2008) 130: 205–250
DOI 10.1007/s10955-007-9434-6

Bulk Universality and Related Properties of Hermitian Matrix Models

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Received: 4 June 2007 / Accepted: 11 September 2007 / Published online: 18 October 2007

Abstract We give a new proof of universality properties in the bulk of spectrum of the hermitian matrix models, assuming that the potential that determines the model is globally C^2 and locally C^3 function (see Theorem 3.1). The proof as our previous proof in (Pastur and Shcherbina in J. Stat. Phys. 86:109–147, 1997) is based on the orthogonal polynomial techniques but does not use asymptotics of orthogonal polynomials. Rather, we obtain the sin-kernel as a unique solution of a certain non-linear integro-differential equation that follows from the determinant formulas for the correlation functions of the model. We also give a simplified and strengthened version of paper (Boutet de Monvel, et al. in J. Stat. Phys. 79:585–611, 1995) on the existence and properties of the limiting Normalized Counting Measure of eigenvalues. We use these results in the proof of universality and we believe that they are of independent interest.

Keywords Matrix model · Local eigenvalue statistics · Universality

1 Introduction

We present an asymptotic analysis of a class of random matrix ensembles, known as matrix models. They are defined by the probability law

$$P_{n,\beta}(d_\beta M) = Z_{n,\beta}^{-1} \exp\left\{-\frac{\beta n}{2} \operatorname{Tr} V(M)\right\} d_\beta M, \tag{1.1}$$

where $M = \{M_{jk}\}_{j,k=1}^n$ is a $n \times n$ real symmetric ($\beta = 1$) or hermitian ($\beta = 2$) matrix, $V : \mathbb{R} \to \mathbb{R}_+$ is a continuous function called the potential of the model and we will assume that

$$V(\lambda) \geq 2(1+\epsilon) \log(1+|\lambda|) \tag{1.2}$$

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for some $\epsilon > 0$,

$$d_1 M = \prod_{1\le j\le k\le n} dM_{jk}, \qquad d_2 M = \prod_{j=1}^{n} dM_{jj} \prod_{j<k} d\Im M_{jk} d\Re M_{jk}, \tag{1.3}$$

and $Z_{n,\beta}$ is the normalization constant.

These ensembles have been actively studied in the last decades because of the number of their interesting properties and applications (see review works [7, 10, 13, 16] and references therein).

The Random Matrix Theory deals with several asymptotic regimes of the eigenvalue distribution, in particular, the global regime, centered around the weak convergence of the Normalized Counting Measure of eigenvalues (see (2.1)), and the local regime, where universality of local eigenvalue statistics is one of the main topics. Universality of various ensembles of hermitian and other matrices has important applications (see [10, 13, 16]) and has been discussed in physics literature since the beginning of modern era of Random Matrix Theory in the early fifties [3, 9, 11, 12, 16–18, 25]. Rigorous proofs of the universality property for the hermitian matrix models ($\beta = 2$) were given in [21] and [6]. Both proofs rely strongly on the orthogonal polynomial techniques, reducing the proof to a certain asymptotic problem (see relation (3.16) below) for a special class of orthogonal polynomials. The reduction is based on remarkable formulas for all marginals of the joint probability density of eigenvalues known as determinant formulas (see formula (2.4) below) for $\beta = 2$.

In this paper we give a new proof of the bulk universality of local regime of hermitian matrix models. The proof is valid for potentials in (1.1) that are of the class C^2 everywhere and have 3 bounded derivatives in a neighborhood of a point, where we prove the universality. We obtain the *sin*-kernel as a unique solution of a certain nonlinear integro-differential equation, while in our previous paper [21] the kernel was obtained, roughly speaking, as a power series in its arguments. Since our proof of universality requires a number of facts on limiting Normalized Counting Measure of eigenvalues of matrix models, the paper includes an updated and simplified version of results of [1] on the existence and properties of the measure. Most of simplifications are possible because of systematic use of book [22].

The paper is organized as follows. In Sect. 2 we treat the global regime and in Sect. 3 the local regime. In the course of our presentation we will need several technical results from [1, 21]. We will give them here (often improving) to make the paper self consistent.

2 Global regime

2.1 Generalities

Denote $\{\lambda_l^{(n)}\}_{l=1}^n$ the eigenvalues of a real symmetric or hermitian matrix M and set for any interval $\varDelta \in \mathbb{R}$

$$N_n(\varDelta) = \sharp\{\lambda_l^{(n)} \in \varDelta, l = 1, \ldots, n\}/n. \tag{2.1}$$

This is the Normalized Counting Measure of eigenvalues of M (empirical distribution in mathematical statistics). In this section we study the convergence of the random measure N_n to a nonrandom limit N which proved to be a probability measure ($N(\mathbb{R}) = 1$) called often the Integrated Density of States.

Our starting point is the joint probability density of eigenvalues, corresponding to (1.1–1.3) [16].

$$p_{n,\beta}(\lambda_1,\dots,\lambda_n)=Q_{n,\beta}^{-1}\exp\left\{-\frac{\beta n}{2}\sum_{i=1}^{n}V(\lambda_i)\right\}|\Delta_n(\Lambda)|^{\beta}, \tag{2.2}$$

where

$$\Delta_n(\Lambda)=\prod_{1\le j<k\le n}(\lambda_i-\lambda_j),\qquad \Lambda=(\lambda_1,\dots,\lambda_n), \tag{2.3}$$

and $Q_{n,\beta}^{-1}$ is the normalization constant.

Let

$$p_{l,\beta}^{(n)}(\lambda_1,\dots,\lambda_l)=\int_{\mathbb{R}^{n-l}}p_{n,\beta}(\lambda_1,\dots,\lambda_l,\lambda_{l+1},\dots,\lambda_n)d\lambda_{l+1}\dots d\lambda_n \tag{2.4}$$

be the lth marginal density of $p_{n,\beta}$. Then, in particular,

$$\overline{N}_n(\Delta):=E\{N_n(\Delta)\}=\int_{\Delta}p_{1,\beta}^{(n)}(\lambda_1)d\lambda_1, \tag{2.5}$$

or

$$\overline{N}_n(\Delta)=\int_{\Delta}\rho_n(\lambda)d\lambda,\qquad \rho_n=p_{1,\beta}^{(n)}. \tag{2.6}$$

The cases $\beta=1$ and $\beta=2$ correspond to real symmetric and hermitian matrices. However, the probability density (2.2) is well defined for any $\beta>0$ (in particular, the case $\beta=4$ corresponds to real quaternion matrices [16]). In this section we will treat the general case of n-independent strictly positive β.

According to Wigner (see [8, 16, 25]) the density (2.2) can be written as the density of the canonical Gibbs measure

$$p_{n,\beta}(\Lambda)=Q_{n,\beta}^{-1}e^{-\beta nH(\Lambda)/2},\qquad \Lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n, \tag{2.7}$$

corresponding to an one-dimensional system of n particles with the Hamiltonian

$$H(\Lambda)=\sum_{i=1}^{n}V(\lambda_i)-\frac{1}{n}\sum_{i<j}\log|\lambda_i-\lambda_j|, \tag{2.8}$$

the temperature $2/\beta n$, and the partition function

$$Q_{n,\beta}=\int_{\mathbb{R}^n}e^{-\beta nH(\Lambda)/2}d\Lambda. \tag{2.9}$$

The first term of the r.h.s. of (2.8) is analogous to the energy of particles due to the external field V and the second term is analogous to the interaction (Coulomb repulsion) energy.

It is important that the Hamiltonian (2.7–2.8) contains the factor $1/n$ in front of the second sum (interaction). This allows us to view (2.7–2.8) as an analog of molecular field models of statistical mechanics. This analogy was implicitly used in physical papers [2, 8, 25]. A rigorous treatment of a rather general class of mean field models was given in [20, 23]. We will use an extension of the treatment to study the limit of NCM (2.1), corresponding to (2.2–2.3). We stress a difference of this problem comparing to that of statistical mechanics.

In the latter the number of particles is explicitly present only in the Hamiltonian (see the factor $1/n$ in the second term of (2.8)), while in the former we have n also in the Gibbs density (2.7). In statistical mechanics terms we have here a mean field model in which the temperature is inverse proportional to the number of particles, while in a standard statistical mechanics treatment the temperature is fixed during the "macroscopic limit" $n \to \infty$. This will imply that the free energy of the model has to be divided by n^2 to have a well defined limit as $n \to \infty$ and that the limit will coincide with the limit as $n \to \infty$ of the ground state energy, also divided by n^2 (see [1, 14] and formulas (2.10–2.11), and (2.28) below).

It is also well known in statistical mechanics that the macroscopic limit of mean field models can be described in terms of certain extremal problems. In our case the problem consists in minimizing the functional

$$\mathcal{E}[m] = \int V(\lambda)m(d\lambda) + \int \log \frac{1}{|\lambda - \mu|} m(d\lambda)m(d\mu) \tag{2.10}$$

defined on the set $\mathcal{M}_1(\mathbb{R})$ of non-negative measures of unit mass (cf. (2.8)).

The variational problem (2.10) goes back to Gauss and is called the minimum energy problem in the external field V. The unit measure N minimizing (2.10) is called the equilibrium measure in the external field V because of its evident electrostatic interpretation as the equilibrium distribution of linear charges on the ideal conductor occupying the axis $\mathbb{R}$ and confined by the external electric field of potential V. We stress that the corresponding variational procedure determine both the (compact) support σ_N of the measure and its form. This should be compared with the widely known variational problem of the theory of logarithmic potential, where the external field is absent but the support is given (see e.g. [15]). The minimum energy problem in the external field (2.10) arises in various domains of analysis and its applications (see [22] for a rather complete account of results and references concerning the problem).

Before to start the systematic exposition we will make notational conventions that will be used everywhere below. First, the integrals without limits will denote the integrals over the whole axis. Second, symbols $C, c, C_1, \dots$ etc. will denote positive finite quantities that do not depend on n and spectral variables and whose value is not important in the corresponding argument.

2.2 Basic Results and Their Proof

We will need certain properties of the variational problem (2.10), given in the following

Proposition 2.1 *Let $V : \mathbb{R} \to \mathbb{R}_+$ be a continuous function satisfying* (1.2). *Then*:

(i) *there exists one and only one measure $N \in \mathcal{M}_1(\mathbb{R})$ such that*

$$\inf_{m \in \mathcal{M}_1(\mathbb{R})} \mathcal{E}[m] = \mathcal{E}[N] > -\infty, \tag{2.11}$$

and N has the finite logarithmic energy

$$\mathcal{L}[N, N] := \int \log \frac{1}{|\lambda - \mu|} N(d\lambda)N(d\mu) < \infty; \tag{2.12}$$

(ii) *the support σ_N of N is compact*;

(iii) *a measure $N \in \mathcal{M}_1(\mathbb{R})$ is as above if and only if the function*

$$u(\lambda; N) = V(\lambda) + 2\int \log \frac{1}{|\lambda - \mu|} N(d\mu) \tag{2.13}$$

satisfies the following relations almost everywhere with respect to N (*in fact except the set of zero capacity*):

$$u(\lambda; N) = u_*, \tag{2.14}$$

where

$$u_* = \inf_{\lambda\in\mathbb{R}} u(\lambda; N) > -\infty; \tag{2.15}$$

(iv) *if the potential* V *satisfies the Hölder condition*

$$|V(\lambda_1) - V(\lambda_2)| \le C(L_1)|\lambda_1 - \lambda_2|^\gamma, \quad |\lambda_{1,2}| \le L_1 \tag{2.16}$$

for some $\gamma > 0$ *and any* $L_1 < \infty$, *then* $u(\lambda; N)$ *also satisfies the Hölder condition with the same* γ:

$$|u(\lambda_1; N) - u(\lambda_2; N)| \le C'(L_1)|\lambda_1 - \lambda_2|^\gamma, \quad |\lambda_{1,2}| \le L_1; \tag{2.17}$$

(v) *if* m *is a finite signed measure of zero charge*, $m(\mathbb{R}) = 0$, *or its support belongs to* $[-1, 1]$, *then*

$$\mathcal{L}[m, m] \ge 0, \tag{2.18}$$

where for any finite signed measures $m_{1,2}$ *we denote*

$$\mathcal{L}[m_1, m_2] = \int \log \frac{1}{|\lambda - \mu|} m_1(d\lambda)m_2(d\mu), \tag{2.19}$$

$\mathcal{L}[m, m] = 0$ *if and only if* $m = 0$, *we have*

$$|\mathcal{L}[m_1, m_2]|^2 \le \mathcal{L}[m_1, m_1]\mathcal{L}[m_2, m_2], \tag{2.20}$$

and (2.19) *defines a Hilbert structure on the space of signed measures with a scalar product* (2.19) *in which the convex cone of non-negative measures such that* $\mathcal{L}[m, m] < \infty$ *is complete, i.e., if* $\{m^{(k)}\}_{k=1}^\infty$ *is a sequence of non-negative measures, satisfying the Cauchy condition with respect to the norm* (2.18), *then there exists a finite non-negative measure* m *such that* $m^{(k)} \to m$ *in the norm* (2.18) *and weakly*;

(vi) *if* $m_{1,2}$ *are finite signed measures with compact supports, and* $m_1(\mathbb{R}) = 0$, *then*

$$\mathcal{L}[m_1, m_2] = \int_0^\infty \frac{\hat{m}_1(p)\hat{m}_2(-p)}{p} dp, \quad \hat{m}_{1,2}(p) = \int e^{ip\lambda} m_{1,2}(d\lambda). \tag{2.21}$$

Proof Assertions (i)–(iii) are proved in Theorem I.1.3 and I.3.3 of [22] for not necessary continuous V, but it is shown there only that $u(\lambda; N)$ satisfies (2.14) almost everywhere with respect to N. We will prove now that if V is continuous, then $u(\lambda; N)$ satisfies condition (2.14) for all $\lambda \in \sigma_N$. To this end consider a point $\lambda_0 \in \mathbb{R}$ such that

$$u(\lambda_0; N) > u_* + \varepsilon, \quad \varepsilon > 0.$$

Since V is continuous, there exists $\delta_1 > 0$ such that

$$V(\lambda) - V(\lambda_0) > -\varepsilon/3, \qquad |\lambda - \lambda_0| \le \delta_1.$$

On the other hand, it is known [15] that for any finite positive measure m the function

$$\mathcal{L}(\lambda; m) = \int \log \frac{1}{|\lambda - \mu|} m(d\mu) \tag{2.22}$$

is upper semicontinuous, i.e. if $\mathcal{L}(\lambda_0; m) < \infty$, then for any $\varepsilon > 0$ there exists $\delta_2 > 0$ such that

$$\mathcal{L}(\lambda; m) > \mathcal{L}(\lambda_0; m) - \varepsilon/3, \qquad |\lambda - \lambda_0| \le \delta_2.$$

Using this property for $m = N$ we obtain from the above inequalities that

$$u(\lambda) > u_* + \varepsilon/3, \qquad |\lambda - \lambda_0| \le \delta := \min\{\delta_1, \delta_2\}.$$

Then (2.14) and (2.15) imply that $N((\lambda_0 - \delta, \lambda_0 + \delta)) = 0$, i.e., $\lambda_0 \notin \sigma_N$. For the case $\mathcal{L}(\lambda_0; N) = \infty$ the proof is the same.

Let us prove assertion (iv) of proposition. It is evident that it suffices to prove that $\mathcal{L}(\lambda; N)$ of (2.22) is a Hölder function in λ. If $\lambda_1, \lambda_2 \in \sigma_N$, then, according to the above $2\mathcal{L}(\lambda_{1,2}; N) = -V(\lambda_{1,2}) + u_*$, and (2.17) follows immediately from (2.16).

Since σ_N is compact, $\mathbb{R} \setminus \sigma_N$ consists of a finite or countable system of open intervals (gaps). Assume that λ_1, λ_2 belong to the same gap $(\lambda_1^*, \lambda_2^*)$: $\lambda_1^* < \lambda_1 < \lambda_2 < \lambda_2^*$. Since $\mathcal{L}''(\lambda; N) > 0$, $\lambda \in (\lambda_1^*, \lambda_2^*)$, and $\mathcal{L}(\lambda; N) \ge (u_* - V(\lambda))/2$, we have

$$\begin{aligned}
&\frac{1}{2}(V(\lambda_1^*) - V(\lambda_1^* + (\lambda_2 - \lambda_1))) \\
&\quad \le \mathcal{L}(\lambda_1^* + (\lambda_2 - \lambda_1); N) - \mathcal{L}(\lambda_1^*; N) \\
&\quad \le \mathcal{L}(\lambda_2; N) - \mathcal{L}(\lambda_1; N) \le \mathcal{L}(\lambda_2^*; N) - \mathcal{L}(\lambda_2^* - (\lambda_2 - \lambda_1); N) \\
&\quad \le \frac{1}{2}(V(\lambda_2^* - (\lambda_2 - \lambda_1)) - V(\lambda_2^*)),
\end{aligned} \tag{2.23}$$

and (2.17) follows from (2.16). Observe now, that this inequality is also valid if $\lambda_1^* = \lambda_1$ or $\lambda_2 = \lambda_2^*$. The case when λ_1 or λ_2 belongs to a semi infinite gap can be studied similarly.

If λ_1, λ_2 belong to different gaps $\lambda_1 \in (\lambda_1^*, \lambda_2^*)$, $\lambda_2 \in (\lambda_3^*, \lambda_4^*)$, then (2.23) implies

$$\begin{aligned}
&|\mathcal{L}(\lambda_1; N) - \mathcal{L}(\lambda_2; N)| \\
&\quad \le |\mathcal{L}(\lambda_1; N) - \mathcal{L}(\lambda_2^*; N)| + |\mathcal{L}(\lambda_2^*; N) - \mathcal{L}(\lambda_3^*; N)| \\
&\qquad + |\mathcal{L}(\lambda_3^*; N) - \mathcal{L}(\lambda_2; N)| \\
&\quad \le C(|\lambda_1 - \lambda_2^*|^\gamma + |\lambda_2^* - \lambda_3^*|^\gamma + |\lambda_3^* - \lambda_2|^\gamma \\
&\quad \le 3^{1-\gamma} C|\lambda_1 - \lambda_2|^\gamma.
\end{aligned}$$

This proves assertion (iv).

Assertion (v) is proved in Theorem 1.16 of [15]. Assertion (vi) is proved in Lemma 6.41 of [4] for the case $m_2(R) = 0$. This implies (2.21) for a general case of m_2. The proposition is proved. □

We formulate now the main result of this section.

Theorem 2.1 *Consider a collection of random variables* $\{\lambda_l^{(n)}\}_{l=1}^n$, *specified by the probability density* (2.2–2.3) *in which* $\beta > 0$ *and the potential* V *satisfies* (1.2) *and* (2.16). *Then*:

(i) *there exists* $0 < L < \infty$ *such that for any* $|\lambda_1|, |\lambda_2| \ge L$

$$\rho_n(\lambda_1) \le e^{-nCV(\lambda_1)}, \qquad p_{2,\beta}^{(n)}(\lambda_1, \lambda_2) \le e^{-nC(V(\lambda_1)+V(\lambda_2))}, \tag{2.24}$$

where ρ_n *and* $p_{2,\beta}^{(n)}$ *are defined in* (2.6) *and* (2.4), *and* L *depends on* ϵ *of* (1.2) *and on*

$$m = \min_{\lambda\in\mathbb{R}}\{V(\lambda) - 2(1+\epsilon/2)\log(1+|\lambda|)\}, \qquad M = \max_{\lambda\in[-1/2,1/2]} V(\lambda); \tag{2.25}$$

(ii) *the Normalized Counting Measure* (2.1) *of the collection* $\{\lambda_l^{(n)}\}_{l=1}^n$ *converges in probability to the unique minimizer* N *of* (2.10–2.11), *and for any differentiable function* $\varphi : [-L, L] \to \mathbb{C}$ *we have*

$$\left|\int \varphi(\mu)\rho_n(\mu)d\mu - \int \varphi(\mu)N(d\mu)\right| \le C\|\varphi'\|_2^{1/2}\|\varphi\|_2^{1/2} \cdot n^{-1/2}\log^{1/2} n, \tag{2.26}$$

$$\left|\int \varphi(\lambda)\varphi(\mu)(p_2^{(n)}(\lambda,\mu) - \rho_n(\lambda)\rho_n(\mu))d\lambda d\mu\right| \le C\|\varphi'\|_2\|\varphi\|_2 \cdot n^{-1}\log n, \tag{2.27}$$

where the symbol $\|\cdots\|_2$ *denotes the* L^2*-norm on* $[-L, L]$;

(iii) *the free energy* $-2(\beta n^2)^{-1}\log Q_{n,\beta}$ *of the model* (2.7–2.9) *converges as* $n \to \infty$ *to the ground state energy* (2.11) *and*

$$\left|\frac{2}{\beta n^2}\log Q_{n,\beta} + \mathcal{E}[N]\right| \le Cn^{-1}\log n. \tag{2.28}$$

Theorem 2.2 *Let* V *satisfy* (1.2) *and* V' *be such that for any* $A > 0$ *there exists* $C(A) > 0$ *providing the inequality*

$$|V'(\lambda) - V'(\mu)| \le C(A)|\lambda - \mu|, \quad |\lambda|, |\mu| \le A. \tag{2.29}$$

Consider the measure N *defined by* (2.11) *and denote* f *its Stieltjes transform*:

$$f(z) = \int \frac{N(d\lambda)}{\lambda - z}, \quad \Im z \ne 0. \tag{2.30}$$

Then f *satisfies the equation*

$$f^2(z) = \int \frac{V'(\lambda)N(d\lambda)}{\lambda - z}, \tag{2.31}$$

N *has a bounded density* ρ *which can be represented in the form*

$$\rho(\lambda) = \frac{1}{2\pi}(4Q(\lambda) - V'^2(\lambda))_+^{1/2}, \tag{2.32}$$

where $(x)_+ = \max\{x, 0\}$,

$$Q(\lambda) = \int \frac{V'(\lambda) - V'(\mu)}{\lambda - \mu}\rho(\mu)d\mu, \tag{2.33}$$

and we have

$$|\rho^2(\lambda) - \rho^2(\mu)| \le C|\lambda - \mu|\log\frac{1}{|\lambda - \mu|}. \tag{2.34}$$

The proof of the theorem is based on the ideas of [19] (see also [5]) and is given below, after the proof of Theorem 2.1.

Remark 2.1 It follows from the theorem that under condition (2.29) we can differentiate the r.h.s. of (2.14) with respect to λ. Then we obtain that ρ solves the singular integral equation

$$V'(\lambda) = 2\int_\sigma \frac{\rho(\mu)d\mu}{\lambda-\mu}, \quad \lambda\in\sigma. \tag{2.35}$$

Theorem 2.3 *Let V satisfy conditions of Theorem 2.2 and $u(\lambda)\neq u_*$ for $\lambda\notin\sigma_N$ (see (2.13), (2.14)). Denote by $\sigma_N^{(\varepsilon)}$ the ε-neighborhood of σ_N and*

$$d_n = \int_{\mathbb{R}\setminus\sigma_N} e^{-\beta n(u(\lambda)-u_*)/4}d\lambda, \qquad d(\varepsilon) = \sup_{\mathbb{R}\setminus\sigma_N^{(\varepsilon)}}\{(u_*-u(\lambda))/4\}. \tag{2.36}$$

Then there exists a n-independent $C>0$ such that for any $\varepsilon>0$ (may be depending on n), satisfying condition $d(\varepsilon) > C(n^{-1/2}\log n + d_n)$ we have the bound (cf. (2.24))

$$\overline{N}_n(\mathbb{R}\setminus\sigma_N^{(\varepsilon)}) \le e^{-nd(\varepsilon)}, \tag{2.37}$$

where $\overline{N}_n$ is defined in (2.5).

The proof of the theorem is given below after the proof of Theorem 2.2.

Remark 2.2 It follows from the proof of the theorem that if we replace (2.29) by conditions (2.16) and $|\sigma_N|\neq 0$, then Theorem 2.3 will also be valid.

Remark 2.3 Usually d_n of (2.36) is $O(n^{-1})$, but it may happen also that $d_n\to 0$ vanishes more slowly as $n\to\infty$.

Proof of Theorem 2.1 Following the main idea of [20, 23] we will use the Bogolyubov inequality (a version of the Jensen inequality) to control the free energy of our "mean field" model. The inequality is given by

Lemma 2.1 *Let $\mathcal{H}_{1,2}:\mathbb{R}^n\to\mathbb{R}$ be such that*

$$Q_{1,2} := \int e^{-\mathcal{H}_{1,2}(\Lambda)/T}d\Lambda < \infty, \quad \Lambda = (\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n,\ T>0.$$

Denote

$$\langle\cdots\rangle_{1,2} = Q_{1,2}^{-1}\int\cdots e^{-\mathcal{H}_{1,2}(\Lambda)/T}d\Lambda.$$

Then

$$\langle\mathcal{H}_1-\mathcal{H}_2\rangle_1 \le T\log Q_2 - T\log Q_1 \le \langle\mathcal{H}_1-\mathcal{H}_2\rangle_2. \tag{2.38}$$

The proof of the lemma is given in the next subsection.

Since the proof of assertion (i) is independent of the proof of (central) assertion (ii), we will give the proof assertion (i) in the next subsection. We will use however assertion (i) in the proof of assertion (ii).

According to assertion (i) the limiting measure N of (2.5), if it exists, has its support strictly inside $[-L, L]$. Let us show that the limiting measure does not depend on values of the potential outside $[-L, L]$. To this end consider potentials V_1 and V_2, satisfying (1.2) and (2.16). Then the potential

$$V(\lambda,t) = tV_1(\lambda) + (1-t)V_2(\lambda) \tag{2.39}$$

also satisfies (1.2) and (2.16). Denote $\overline{N}_n(\cdot,t)$, $\rho_n(\cdot,t)$, and $p_{2,\beta}^{(n)}(\cdot,\cdot,t)$ the measure (2.6), its density, and the second marginal of (2.2) corresponding to (2.39). Then it is easy to find from (2.2–2.4) that

$$\begin{aligned}\frac{\partial}{\partial t}\rho_n(\lambda,t) &= -n\delta V(\lambda)\rho_n(\lambda,t) - n(n-1)\int \delta V(\mu)p_{2,\beta}^{(n)}(\lambda,\mu,t)d\mu \\ &\quad + n^2\rho_n(\lambda,t)\int \delta V(\mu)\rho_n(\mu,t)d\mu,\end{aligned} \tag{2.40}$$

where $\delta V = V_1 - V_2$. This implies the bound

$$\left|\frac{\partial}{\partial t}\overline{N}_n(\Delta,t)\right| \le 2n^2\int |\delta V(\mu)|\rho_n(\mu,t)d\mu,$$

valid for any $\Delta \in \mathbb{R}$. Now, if $V_1(\lambda) = V_2(\lambda)$, $|\lambda| < L$, then in view of (2.24) and (1.2) we have:

$$\begin{aligned}&|\overline{N}_n(\Delta)|_{V=V_1} - \overline{N}_n(\Delta)|_{V=V_2}| \\ &\quad\le 2n^2\int_{|\lambda|>L} d\lambda|V_1(\lambda) - V_2(\lambda)|\int_0^1 e^{-nCV(\lambda,t)}dt \\ &\quad\le 2C^{-1}n\int_{|\lambda|>L}(e^{-nCV_1(\lambda)} + e^{-nCV_2(\lambda)})d\lambda = O(e^{-nC'}).\end{aligned} \tag{2.41}$$

We conclude that without loss of generality we can assume that the potential satisfies the Hölder condition on the whole axis with the same exponent as in (2.16):

$$|V(\lambda_1) - V(\lambda_2)| \le C|\lambda_1 - \lambda_2|^\gamma, \quad \lambda_1,\lambda_2 \in \mathbb{R}. \tag{2.42}$$

Furthermore, we can also assume without loss of generality that the parameter L of assertion (i) of the theorem is less than $1/2$ and that the support σ_N of the minimizer N of (2.10–2.11) and all the points λ_k^* such that $u(\lambda_k^*) = u_*$ are contained in the interval $[-1/2+\delta, 1/2-\delta]$ for some $\delta > 0$.

Let us prove (2.26). Denote by $\mathcal{C}^*$ the cone of measures on $\mathbb{R}$ satisfying the conditions:

$$m(d\lambda) \ge 0, \qquad \operatorname{supp} m \subset [-1/2, 1/2], \qquad \mathcal{L}[m,m] < \infty, \qquad m(\mathbb{R}) \le 1. \tag{2.43}$$

For any $m \in \mathcal{C}^*$ we introduce the "approximating" Hamiltonian

$$H_a(\Lambda; m) = \sum_{i=1}^n u_n(\lambda_i; m) - (n-1)\mathcal{L}[m,m], \tag{2.44}$$

where (cf. (2.13))

$$u_n(\lambda; m) = V(\lambda) + 2\frac{n-1}{n}\mathcal{L}(\lambda; m), \tag{2.45}$$

and $\mathcal{L}(\lambda; m)$, $\mathcal{L}[m, m]$ are defined by (2.22) and (2.19). Consider the functional $\Phi_n : \mathcal{C}^* \to \mathbb{R}$, defined as

$$\begin{aligned}\Phi_n[m] &= \frac{2}{\beta n^2} \log \int e^{-\beta n H_a(\Lambda; m)/2} d\Lambda \\ &= \frac{(n-1)}{n} \mathcal{L}[m, m] + \frac{2}{\beta n} \log \int e^{-\beta n u_n(\lambda; m)/2} d\lambda. \end{aligned} \tag{2.46}$$

Taking in (2.38) $\mathcal{H}_1 = H$, $\mathcal{H}_2 = H_a$ and $T = 2/\beta n$, we obtain

$$R[m] \le \Phi_n[m] - \frac{2}{\beta n^2} \log Q_{n,\beta} \le R_a[m], \tag{2.47}$$

where

$$R[m] = 2(\beta n^2 Q_{n,\beta})^{-1} \int (H - H_a) e^{-\beta n H/2} d\Lambda,$$

$$R_a[m] = 2(\beta n^2)^{-1} e^{-\beta n^2 \Phi_n[m]/2} \int (H - H_a) e^{-\beta n H_a(\Lambda; m)/2} d\Lambda,$$

and $Q_{n,\beta}$ is defined in (2.9). Since H and H_a are symmetric, we can rewrite $R[m]$ as follows

$$\begin{aligned} R[m] = \frac{n-1}{n} \bigg(\int \log \frac{1}{|\lambda - \mu|} (p_{2,\beta}^{(n)}(\lambda, \mu) - \rho_n(\lambda)\rho_n(\mu)) d\lambda d\mu \\ + \mathcal{L}[\overline{N}_n - m, \overline{N}_n - m] \bigg), \end{aligned} \tag{2.48}$$

where $p_{2,\beta}^{(n)}$, ρ_n, and $\overline{N}_n$ are defined in (2.4), (2.5–2.6). To obtain R_a, we have to replace $\rho_n(\lambda)$ and $p_{2,\beta}^{(n)}(\lambda, \mu)$ in (2.48) by $\rho_n^{(a)}(\lambda; m)$ and $\rho_n^{(a)}(\lambda; m)\rho_n^{(a)}(\mu; m)$, the correlation functions of the approximating Hamiltonian (2.44), where

$$\rho_n^{(a)}(\lambda; m) = e^{-\beta n u_n(\lambda; m)/2} \bigg(\int d\lambda e^{-\beta n u_n(\lambda; m)/2} \bigg)^{-1}. \tag{2.49}$$

This yields:

$$R_a[m] = \frac{n-1}{n} \mathcal{L}[N_n^{(a)} - m, N_n^{(a)} - m], \tag{2.50}$$

where

$$N_n^{(a)}(d\lambda; m) = \rho_n^{(a)}(\lambda; m) d\lambda. \tag{2.51}$$

Lemma 2.2 *Let $\mathcal{C}^*$ be the cone of measures defined by* (2.43) *and the functional $\Phi_n : \mathcal{C}^* \to \mathbb{R}$ be given by* (2.46). *Then*

(i) *Φ_n attains its minimum at a unique point $m_n \in \mathcal{C}^*$ and*

$$\mathcal{L}[N_n^{(a)} - m_n, N_n^{(a)} - m_n] \le e^{-nc}; \tag{2.52}$$

(ii) *if N is a measure, defined by* (2.8–2.13), *then*

$$0 \le \Phi_n[N] - \Phi_n[m_n] \le C n^{-1} \log n. \tag{2.53}$$

The proof of Lemma 2.2 is given in the next subsection.

On the basis of (2.47), Lemma 2.2, and (2.50) we obtain

$$\begin{aligned}R[N] &\le \Phi_n[N] - \frac{2}{\beta n^2}\log Q_{n,\beta}\\ &= (\Phi_n[N]-\Phi_n[m_n]) + \left(\Phi_n[m_n] - \frac{2}{\beta n^2}\log Q_{n,\beta}\right)\\ &\le Cn^{-1}\log n + R_a[m_n] \le Cn^{-1}\log n + Ce^{-nc}. \end{aligned}\tag{2.54}$$

This and (2.48) lead to the inequality

$$\int \log\frac{1}{|\lambda-\mu|}(p_2^{(n)}(\lambda,\mu)-\rho_n(\lambda)\rho_n(\mu))d\lambda d\mu + \mathcal{L}[\overline{N}_n - N, \overline{N}_n - N] \le Cn^{-1}\log n. \tag{2.55}$$

Since $\mathcal{L}[\overline{N}_n - N, \overline{N}_n - N] \ge 0$ by Proposition 2.1 (v), we have the bound

$$\begin{aligned}&\int \log\frac{1}{|\lambda-\mu|}G_n(\lambda,\mu)d\lambda d\mu \le C\frac{\log n}{n},\\ &G_n(\lambda,\mu) = p_2^{(n)}(\lambda,\mu) - \rho_n(\lambda)\rho_n(\mu).\end{aligned}\tag{2.56}$$

We will prove now that there exists a n-independent $C>0$ such that

$$\int \log\frac{1}{|\lambda-\mu|}G_n(\lambda,\mu)d\lambda d\mu, \ge -C\frac{\log n}{n}, \tag{2.57}$$

and, as a result, that

$$\int \log\frac{1}{|\lambda-\mu|}G_n(\lambda,\mu)d\lambda d\mu = O(n^{-1}\log n). \tag{2.58}$$

Note that (2.58) and (2.54) yield assertion (iii) of Theorem 2.1. Indeed, it follows from (2.55) and (2.58) that

$$\mathcal{L}[\overline{N}_n - N, \overline{N}_n - N] = O(n^{-1}\log n). \tag{2.59}$$

This and (2.54) imply

$$\Phi_n[N] - \frac{2}{\beta n^2}\log Q_{n,\beta} = O(n^{-1}\log n). \tag{2.60}$$

Since according to (2.17) $\mathcal{L}(\lambda; N)$ is a Hölder function, it is easy to find by the Laplace method that

$$\begin{aligned}\Phi_n[N] &= \frac{n-1}{n}\mathcal{L}[N;N] - \min_{\lambda}\{u(\lambda;N)\} + O(n^{-1}\log n)\\ &= \frac{n-1}{n}\mathcal{L}[N;N] - \int u(\lambda;N)N(d\lambda) + O(n^{-1}\log n)\\ &= -\mathcal{E}[N] + O(n^{-1}\log n).\end{aligned}$$

Here $u(\lambda; N)$ is defined by (2.13) and we have used (2.14). The two last relations yield (2.28).

To prove (2.57) we need certain upper bounds for ρ_n and $p_2^{(n)}$. Changing variables $\lambda_i \to \lambda_i - x$ and using (2.42) we find that for any $|x| \le h := n^{-3/\gamma}$

$$\begin{aligned} &|\rho_n(\lambda_1 + x) - \rho_n(\lambda_1)| \\ &\quad = Q_{n,\beta}^{-1} \left| \int d\lambda_2 \ldots d\lambda_n \cdot |\Delta(\Lambda)|^\beta \times e^{-nV(\lambda_1+x)} \prod_{i=2}^n e^{-nV(\lambda_i - x)} - \prod_{i=2}^n e^{-nV(\lambda_i)} \right| \\ &\quad \le C n^2 x^\gamma \rho_n(\lambda_1). \end{aligned} \tag{2.61}$$

Now we use the simple identity valid for any interval $[a, b]$ and any integrable function $v(\lambda)$

$$v(\lambda) = (b-a)^{-1} \int_a^b (v(\lambda) - v(\mu)) d\mu + (b-a)^{-1} \int_a^b v(\mu) d\mu. \tag{2.62}$$

The identity with $v(\lambda) = \rho_n(\lambda)$, $a = \lambda$, $b = \lambda + h$, (2.61), and the normalization condition

$$\int \rho_n(\lambda) d\lambda = 1 \tag{2.63}$$

lead to the inequality

$$\rho_n(\lambda) \le C n^{-1} \rho_n(\lambda) + n^{3/\gamma},$$

implying

$$\rho_n(\lambda) \le C n^{3/\gamma}. \tag{2.64}$$

Similarly we have for $p_2^{(n)}$ of (2.4), and G_n of (2.56):

$$p_2^{(n)}(\lambda, \mu) \le C n^{6/\gamma}, \qquad \int G_n^2(\lambda, \mu) d\lambda d\mu \le C n^{6/\gamma}. \tag{2.65}$$

Furthermore, we can write the equality

$$\log |t|^{-1} = \sum_{k=-\infty}^{\infty} l^{(k)} e^{ikt\pi}, \quad |t| \le 1, \tag{2.66}$$

valid in $L^2([-1, 1])$ and in which

$$\frac{C_2}{|k|} < l^{(k)} = \frac{1}{\pi |k|} \int_0^{\pi |k|} \frac{\sin x}{x} dx < \frac{C_1}{|k|}, \quad k \ne 0. \tag{2.67}$$

Besides, since for any bounded continuous function $f : \mathbb{R} \to \mathbb{C}$ we have

$$\int \left| \frac{1}{n} \sum_{i=1}^n (f(\lambda_i) - \langle f \rangle) \right|^2 p_n(\Lambda) d\Lambda \ge 0, \quad \langle f \rangle = \int f(\lambda) \rho_n(\lambda) d\lambda,$$

the symmetry of p_n of (2.2) implies:

$$\int f(\lambda) \overline{f}(\mu) G_n(\lambda, \mu) d\lambda d\mu + (n-1)^{-1} \langle |f|^2 \rangle \ge 0. \tag{2.68}$$

We now write integral in (2.57) as that over the square $\{|\lambda| \le 1/2, |\mu| \le 1/2\}$ and over the complement of the square. The second integral is $O(e^{-nc})$ by (2.24) and (2.65). In the first integral we replace $\log|\lambda-\mu|^{-1}$ by the r.h.s. of (2.66) with $t=\lambda-\mu$. Thus, choosing $M=n^{2+6/\gamma}$, we get:

$$\begin{aligned}\int \log\frac{1}{|\lambda-\mu|}G_n(\lambda,\mu)d\lambda d\mu &= O(e^{-nc})+\sum_{k\neq 0}G_n^{(k,k)}l^{(k)}\\ &=\sum_{k<M}(G_n^{(k,k)}+(n-1)^{-1})l^{(k)}-(n-1)^{-1}\sum_{k<M}l^{(k)}+R_M\\ &\ge -Cn^{-1}\log n+R_M.\end{aligned}\tag{2.69}$$

Here

$$G_n^{(k,m)}=\int_{-1/2}^{1/2}e^{ik\pi\lambda-im\pi\mu}G_n(\lambda,\mu)d\lambda d\mu,$$

and we use (2.24) implying $G_n^{(0,0)}=O(e^{-Cn})$ and (2.68) implying $G_n^{(k,k)}+n^{-1}\ge 0$. To estimate R_M we use (2.67) and (2.65) to write

$$|R_M|\le\sum_{k>M}|G_n^{(k,k)}l^{(k)}|\le\left[\sum_{k,m}|G_n^{(k,m)}|^2\right]^{1/2}\left[\sum_{k>M}|l^{(k)}|^2\right]^{1/2}\le C\frac{n^{3/\gamma}}{M^{1/2}}.\tag{2.70}$$

The bound (2.58) follows from (2.69) and (2.70).

Consider now a function $\varphi:[-1/2,1/2]\to\mathbb{C}$ such that $\varphi'\in L^2[-1/2,1/2]$ and denote

$$\begin{aligned}\varphi_1(\lambda)&=\varphi(\lambda)\mathbf{1}_{|\lambda|\le 1/2}+2\varphi(-1/2)(1+\lambda)\mathbf{1}_{\lambda<-1/2}+2\varphi(1/2)(1-\lambda)\mathbf{1}_{\lambda>1/2},\\ \varphi^{(k)}&=\frac{1}{2}\int_{-1}^{1}\varphi_1(\lambda)e^{ik\pi\lambda}d\lambda,\\ d^{(k)}&=\frac{1}{2}\int_{-1}^{1}e^{ik\pi\lambda}(N(d\lambda)-\overline{N}_n(d\lambda)).\end{aligned}$$

Then, using (2.24), (2.67), and the Parseval equation, we get

$$\begin{aligned}&\left|\int\varphi(\lambda)(N(d\lambda)-\overline{N}_n(d\lambda))\right|^2\\ &\quad=\left|O(e^{-nc})+\sum_{k\in\mathbb{Z}}\varphi^{(k)}d^{(k)}\right|^2\\ &\quad\le C\sum_{k\in\mathbb{Z}}l^{(k)}|d^{(k)}|^2\sum_{k\in\mathbb{Z}}|k||\varphi^{(k)}|^2+O(e^{-nc})\\ &\quad\le C\mathcal{L}[N-\overline{N}_n,N-\overline{N}_n]\cdot\|\varphi\|_2\|\varphi'\|_2+O(e^{-nc}).\end{aligned}$$

This inequality and (2.59) imply (2.26). The inequality (2.27) can be proved similarly.

We will prove now that for any finite interval $\varDelta\subset\mathbb{R}$ $N_n(\varDelta)$ of (2.1) converges in probability to $N(\varDelta)$ of (2.11) as $n\to\infty$, i.e. that for any $\varepsilon>0$

$$\lim_{n\to\infty}\mathbf{P}\{|N(\varDelta)-N_n(\varDelta)|>\varepsilon\}=0,$$

where $\mathbf{P}\{\cdots\}$ denotes the probability, corresponding to (2.2). This is the first part of assertion (ii) of Theorem 2.1.

Denote $\Delta = (a, b)$, $-\infty < a < b < \infty$, χ the indicator of Δ, and χ_+ the continuous function, coinciding with χ on (a, b), equal zero outside $(a-\delta, b+\delta)$ for a sufficiently small δ and linear on $(a-\delta, a)$ and $(b, b+\delta)$. Let χ_- be the analogous function for the interval $(a+\delta, b-\delta)$. Then

$$\chi_- \le \chi \le \chi_+, \qquad \|\chi_\pm\|_2^2 \le b-a+\delta, \qquad \|\chi_\pm'\|_2^2 \le 2\delta^{-1}. \tag{2.71}$$

Hence

$$N_n[\chi_-] \le N_n[\chi] \le N_n[\chi_+], \tag{2.72}$$

where we denote for any $\varphi : \mathbb{R} \to \mathbb{C}$

$$N_n[\varphi] = n^{-1}\sum_{l=1}^{n} \varphi(\lambda_l^{(n)}) = \int \varphi(\lambda) N_n(d\lambda). \tag{2.73}$$

This is a normalized linear statistics of random variables $\{\lambda_l^{(n)}\}_{l=1}^n$, corresponding to the test function φ. We have in particular $N_n[\chi] = N_n(\Delta)$. By using this notation, we can rewrite (2.26) as

$$|\mathbf{E}\{N_n[\varphi]\} - N[\varphi]| \le Cn^{-1/2}\log^{1/2} n \|\varphi\|_2^{1/2}\|\varphi'\|_2^{1/2}, \tag{2.74}$$

where $\mathbf{E}\{\cdots\}$ denotes the expectation with respect to (2.2) and

$$N[\varphi] = \int \varphi(\lambda) N(d\lambda).$$

Choosing in (2.74) $\varphi = \chi_\pm$, taking into account (2.71) and the continuity of N and making first the limit $n \to \infty$ and then $\delta \to 0$, we obtain

$$\lim_{n\to\infty} \mathbf{E}\{N_n(\Delta)\} = N(\Delta). \tag{2.75}$$

Likewise, denoting

$$\mathbf{Var}\{N_n[\varphi]\} = \mathbf{E}\{N_n^2[\varphi]\} - \mathbf{E}^2\{N_n[\varphi]\},$$

we obtain from (2.27)

$$\mathbf{Var}\{N_n[\varphi]\} \le Cn^{-1}\log n \|\varphi\|_2\|\varphi'\|_2. \tag{2.76}$$

Using this bound, (2.71), and (2.72) we obtain

$$\lim_{n\to\infty} \mathbf{Var}\{N_n(\Delta)\} = 0. \tag{2.77}$$

Formulas (2.75) and (2.77) imply the convergence of the sequence $\{N_n[\Delta]\}$ in probability to the nonrandom limit $N(\Delta)$ for any finite $\Delta \subset \mathbb{R}$. Theorem 2.1 is proved. □

Remark 2.4 Inspecting the above proof of Theorem 2.1, we conclude that its assertions remain valid if we replace the potential V in (2.2) by $V + \varepsilon_n V_1$, where V_1 satisfies (1.2) and (2.16) and $\varepsilon_n = O(n^{-1}\log n)$. If $\varepsilon_n \to 0$ more slowly, then $n^{-1/2}\log^{-1/2} n$ and $n^{-1}\log^{-1} n$ in the r.h.s. of (2.26) and (2.27) should be replaced by $\varepsilon_n^{1/2}$ and ε_n respectively.

Proof of Theorem 2.2 We follow the idea of [19] (see also [5]). Consider a collection of random variables $\{\lambda_l^{(n)}\}_{l=1}^n$, specified by the probability density (2.2–2.3) for $\beta = 2$. We remark first that without loss of generality we can assume that $V(\lambda)$ is a linear function outside of the interval $[-L, L]$, where L is defined in assertion (i) of Theorem 2.1 and hence, in view of (2.29), that

$$\sup_{\lambda\in\mathbb{R}} |V'(\lambda)| \le C < \infty. \tag{2.78}$$

Indeed, it suffices to repeat the argument, leading to (2.42).

We have from (2.2–2.4) for $\beta = 2$ and $l = 1$:

$$\rho_n(\lambda) = Q_{n,2}^{-1} \int e^{-nV(\lambda)} \prod_{j=2}^{n} d\lambda_j e^{-nV(\lambda_j)} (\lambda - \lambda_j)^2 \prod_{2\le j<k\le n} (\lambda_j - \lambda_k)^2. \tag{2.79}$$

Then, taking any z with $\Im z \neq 0$ and integrating by parts, we obtain from (2.79) that

$$\int \frac{V'(\lambda)\rho_n(\lambda)}{\lambda - z} d\lambda = -\frac{1}{n} \int \frac{\rho_n(\lambda)}{(\lambda - z)^2} d\lambda + \frac{2(n-1)}{n} \int \frac{p_2^{(n)}(\lambda,\mu) d\lambda d\mu}{(\lambda-\mu)(\lambda - z)}. \tag{2.80}$$

Since $p_2^{(n)}(\lambda,\mu) = p_2^{(n)}(\mu,\lambda)$, we have

$$2\int \frac{p_2^{(n)}(\lambda,\mu) d\lambda d\mu}{(\lambda-\mu)(\lambda - z)} = -\int \frac{p_2^{(n)}(\lambda,\mu) d\lambda d\mu}{(\lambda-z)(\mu - z)},$$

and (2.80) takes the form

$$\int \frac{V'(\lambda)\rho_n(\lambda)}{\lambda - z} d\lambda = -\frac{1}{n} \int \frac{\rho_n(\lambda)}{(\lambda - z)^2} d\lambda - \frac{n-1}{n} \int \frac{G_n(\lambda,\mu)}{(\mu-z)(\lambda - z)} d\lambda d\mu$$
$$- \frac{n-1}{n} \left(\int \frac{\rho_n(\lambda)}{\lambda - z} d\lambda \right)^2, \tag{2.81}$$

where $G_n(\lambda,\mu)$ was defined in (2.56). Thus, denoting

$$f_n(z) = \int \frac{\rho_n(\lambda) d\lambda}{\lambda - z} \tag{2.82}$$

the Stieltjes transform of ρ_n, we derive from (2.81) for $z = \lambda_0 + i\eta$, $\eta \neq 0$:

$$\frac{n-1}{n} f_n^2(z) + \int \frac{V'(\lambda)\rho_n(\lambda)}{\lambda - z} d\lambda = -\frac{1}{n} \int \frac{\rho_n(\lambda) d\lambda}{(\lambda - z)^2} - \frac{n-1}{n} \int \frac{G_n(\lambda,\mu) d\lambda d\mu}{(\mu-z)(\lambda - z)},$$

and the second integral in the l.h.s. is well defined, since V is linear for large absolute values of its argument (see the beginning of proof of the theorem). Moreover, this and (2.26) allow us to pass to the limit $n \to \infty$ in this term. The first term in the r.h.s. of (2.83) is $O(n^{-1})$ for any fixed z, $\Im z \neq 0$. According to (2.27) the second term in the r.h.s. of (2.83) also vanishes in the limit $n \to \infty$ and, according to (2.26), $f_n(z) \to f(z)$ as $n \to \infty$ uniformly on a compact set of $\mathbb{C} \setminus \mathbb{R}$. Therefore, taking the limit $n \to \infty$ in (2.83), we get equation (2.31). Setting $z = \lambda + i\eta$, we rewrite the equation as

$$f^2(z) + V'(\lambda) f(z) + \int \frac{V'(\mu) - V'(\lambda)}{\mu - z} N(d\mu) = 0. \tag{2.83}$$

Solving this quadratic equation in f and using the inversion formula for the Stieltjes transform, we obtain (2.32–2.33).

Note that (2.32) and (2.29) imply that $\rho(\lambda)$ is bounded, because

$$|Q(\lambda)| \le \int \frac{|V'(\lambda)-V'(\mu)|}{|\lambda-\mu|}\rho(\mu)d\mu \le C\int \rho(\mu)d\mu = C.$$

It is also clear from (2.29) and (2.32) that to prove (2.34) it suffices to prove the same inequality for $Q(\lambda)$. To this end we take any $h>0$ and write

$$\begin{aligned}
&|Q(\lambda+h)-Q(\lambda)| \\
&\quad\le \int_{|\lambda-\mu|\le 2h}\left(\frac{|V'(\lambda)-V'(\mu)|}{|\lambda-\mu|}+\frac{|V'(\lambda+h)-V'(\mu)|}{|\lambda+h-\mu|}\right)\rho(\mu)d\mu \\
&\qquad + \int_{|\lambda-\mu|>2h}\left(\frac{|V'(\lambda+h)-V'(\lambda)|}{|\lambda+h-\mu|}+\frac{|V'(\lambda)-V'(\mu)|h}{|\lambda-\mu|\cdot|\lambda+h-\mu|}\right)\rho(\mu)d\mu \\
&\quad\le C\sup_{\lambda\in\mathbb{R}}\rho(\lambda)h(|\log h|+1).
\end{aligned}$$

Theorem 2.2 is proved. □

Proof of Theorem 2.3 Set

$$V_1(\lambda)=\frac{1}{2}(u(\lambda;N)-u_*), \qquad u_1(\lambda)=u(\lambda;N)-V_1(\lambda), \tag{2.84}$$

where $u(\lambda;N)$ and u_* are defined by (2.13–2.14). It is easy to see that $V_1(\lambda)=0$, $\lambda\in\sigma_N$, $V_1(\lambda)\ge 0$, $\lambda\notin\sigma_N$, and $u_1(\lambda)$ attains its minimum u_* for $\lambda\in\sigma_N$.

Consider the Hamiltonians:

$$\begin{aligned}
H_n^{(1)}(\Lambda) &= -V_1(\lambda_1)+\sum_{i=1}^n V(\lambda_i)-\frac{2}{n}\sum_{1\le i<j\le n}\log|\lambda_i-\lambda_j|, \\
H_n^{(1a)}(\Lambda) &= \frac{n-1}{n}u_1(\lambda_1)+\frac{1}{n}(V(\lambda_1)-V_1(\lambda_1)) \\
&\qquad +\sum_{i=2}^n V(\lambda_i)-\frac{2}{n}\sum_{2\le i<j\le n}\log|\lambda_i-\lambda_j|.
\end{aligned} \tag{2.85}$$

Denote

$$p^{\natural}_{n,\beta}(\Lambda)=(Q^{\natural}_{n,\beta})^{-1}\exp\{-\beta n H_n^{\natural}(\Lambda)/2\}, \quad \natural=(1),(1a)$$

the corresponding probability densities (cf. (2.2)).

Using the r.h.s inequality in (2.38) for $\mathcal{H}_1=H_n^{(1)}$, $\mathcal{H}_2=H_n^{(1a)}$ and $T=2/\beta n$, we get

$$\log Q^{(1)}_{n,\beta}-\log Q^{(1a)}_{n,\beta}\le I_1+I_2, \tag{2.86}$$

where

$$I_1 = \beta \sum_{i=2}^{n} \int \log|\lambda_1 - \lambda_i|(p_2^{(n,1)}(\lambda_1, \lambda_i) - \rho_n^{(1)}(\lambda_1)\rho_n^{(2)}(\lambda_i))d\lambda_1 d\lambda_i,$$

$$I_2 = \beta(n-1)\int \log|\lambda_1 - \lambda_2|(\overline{N}_n^{(2)}(d\lambda_2) - N(d\lambda_2))\rho_n^{(1)}(\lambda_1)d\lambda_1,$$

$\rho_n^{(1)}$, and $\rho_n^{(2)}$ are the first marginal densities corresponding to λ_1 and λ_i, $i = 2, \dots, n$ for the Hamiltonian $H_n^{(1)}$, $\overline{N}_n^{(\alpha)}(d\lambda) = \rho_n^{(\alpha)}(\lambda)d\lambda$, $\alpha = 1, 2$ (note that $\rho_n^{(1)} \neq \rho_n^{(2)}$ since $H_n^{(1)}$ is not symmetric in λ_1 and $\lambda_i, i = 2, \dots, n$), $p_2^{(n,1)}$ and $p_2^{(n,2)}$ are the second marginal densities, corresponding to $\lambda_1, \lambda_i, i = 2, \dots, n$ and $\lambda_i, \lambda_j, 2 \le i < j \le n$ (note that $p_2^{(n,1)} \neq p_2^{(n,2)}$ and $p_2^{(n,1)}$ is not symmetric because of the same reason).

Repeating the argument that leads to formulas (2.96) and (2.97) below, we get analogs of (2.24) for $\rho_n^{(1)}$ and $\rho_n^{(1)}$ that allow us to restrict integration in the r.h.s. of (2.86) to $[-1/2, 1/2]$. Besides, using (2.66) for $\log|\lambda - \mu|^{-1}$ we obtain similarly to (2.69) and (2.70)

$$\begin{aligned}
|I_1| &\le O(e^{-nc}) + \beta\left|\sum_{|k|<M} l^{(k)}\left\langle e^{ikr\lambda_1}\sum_{j=2}^{n}(e^{ik\pi\lambda_j} - \langle e^{ikr\lambda_j}\rangle)\right\rangle\right| + |R_M| \\
&\le \beta\left[\sum_{|k|<M} l^{(k)}\right]^{1/2}\left[\sum_{|k|<M} l^{(k)}\left\langle\left|\sum_{j=2}^{n}(e^{ik\pi\lambda_j} - \langle e^{ik\pi\lambda_j}\rangle)\right|^2\right\rangle\right]^{1/2} + |R_M| \\
&\le C\log^{1/2} M\left[O(\log^{1/2} M) + n^2\int_{-1}^{1}\log\frac{1}{|\lambda-\mu|}G_n^{(2)}(\lambda,\mu)d\lambda d\mu - R_M^{(1)}\right]^{1/2}, \quad (2.87)
\end{aligned}$$

where we denote (cf. (2.56))

$$\langle\cdots\rangle := \int(\cdots)p_{n,\beta}^{(1)}(\Lambda)d\Lambda, \qquad G_n^{(2)}(\lambda,\mu) = p_2^{(n,2)}(\lambda,\mu) - \rho_n^{(2)}(\lambda)\rho_n^{(2)}(\mu),$$

$M = n^{2+6/\gamma}$ and R_M and $R_M^{(1)}$ are the remainder terms which are the contributions of sums from $|j| = M + 1$ to ∞ in the Fourier series (see (2.70) for the estimate of such terms).

Likewise, considering $H_a^{(1)}$ of the form (2.44) with $V(\lambda_1)$ replaced by $V(\lambda_1) - V_1(\lambda_1)$ and repeating the arguments, leading to (2.58) and (2.59), we obtain analogs of these inequalities for the Hamiltonian $H_n^{(1)}$:

$$\begin{aligned}
&\int\log\frac{1}{|\lambda-\mu|}G_n^{(2)}(\lambda,\mu)d\lambda d\mu = O\left(\frac{\log n}{n}\right), \\
&0 \le \mathcal{L}[\overline{N}_n^{(2)} - N, \overline{N}_n^{(2)} - N] \le \frac{C\log n}{n}.
\end{aligned} \quad (2.88)$$

This and (2.87) yield $I_1 = O(n^{1/2}\log n)$. Similarly, on the basis of the second line of (2.88) and the Schwarz inequality we get $I_2 = O(n^{1/2}\log n)$. Plugging these estimates in (2.86), we obtain

$$\log Q_{n,\beta}^{(1)} - \log Q_{n,\beta}^{(1a)} \le Cn^{1/2}\log n. \quad (2.89)$$

Now we use the r.h.s inequality in (2.38) for $\mathcal{H}_1 = H_n^{(1a)}$, $\mathcal{H}_2 = H_n$ and $T = 2/\beta n$ to get the bound

$$\begin{aligned}\log Q^{(1a)}_{n,\beta} &- \log Q_{n,\beta}\\ &\le \beta n\int V_1(\lambda)\rho_n^{(1a)}(\lambda)d\lambda\\ &\quad+\beta(n-1)\int \log|\lambda_1-\lambda_2|(\rho_n^{(2a)}(\lambda_2)d\lambda_2-N(d\lambda_2))\rho_n^{(1a)}(\lambda_1)d\lambda_1,\end{aligned}\tag{2.90}$$

where $\rho_n^{(1a)}$ and $\rho_n^{(2a)}$ are the first marginal densities of the Hamiltonian $H_n^{(1a)}$, corresponding to λ_1 and λ_i, $i=2,\dots,n$. It is easy to see that (cf. (2.49))

$$\rho_n^{(1a)}(\lambda)=\frac{\exp\{\beta[-(n-1)u_1(\lambda)/2+V_1(\lambda)-V(\lambda)]\}}{\int\exp\{\beta[-(n-1)u_1(\lambda)/2+V_1(\lambda)-V(\lambda)]\}d\lambda}.\tag{2.91}$$

According to definitions (2.84) and (2.13) $V_1(\lambda)=0$ for $\lambda\in\sigma_N$ and in view of (1.2) and Proposition 2.1 (see (2.17)), the function V_1-V admits the bounds:

$$\begin{aligned}V_1(\lambda)-V(\lambda)&\le C,\quad \lambda\in\mathbb{R},\\ V_1(\lambda)-V(\lambda)&\ge -C,\quad \lambda\in\sigma_N.\end{aligned}$$

Besides, the integral in the denominator of (2.91) is bounded from below by the integral over σ_N, which is bounded from below by $|\sigma_N|\exp\{-\beta(n-1)u_*/2-C\}$ and according to Theorem 2.2 $|\sigma_N|\ne 0$, where $|\sigma_N|$ is the Lebesgue measure of σ_N. Taking into account the above bounds, and denoting I_1' the first term in the r.h.s. of (2.90), we obtain

$$|I_1'|\le e^{2C}d_n/|\sigma_N|,$$

where d_n is defined in (2.36).

The second term in the r.h.s. of (2.90) can be estimated by Schwarz inequality (2.20):

$$\begin{aligned}\int &\log|\lambda_1-\lambda_2|(\rho_n^{(2a)}(\lambda_2)d\lambda_2-N(d\lambda_2))\rho_n^{(1a)}(\lambda_1)d\lambda_1\\ &=-\mathcal{L}[\rho_n^{(2a)}d\lambda-N,\rho_n^{(1a)}d\lambda]\\ &\le\mathcal{L}^{1/2}[\rho_n^{(2a)}d\lambda-N,\rho_n^{(2a)}d\lambda-N]\mathcal{L}^{1/2}[\rho_n^{(1a)}d\lambda,\rho_n^{(1a)}d\lambda].\end{aligned}$$

According to the above $\rho_n^{(1a)}$ is bounded and decays at infinity as $C_1\exp\{-nC_2V(\lambda)\}$, hence the second factor is bounded. To estimate the first factor we note that $\rho_n^{(2a)}$ coincides with the first marginal density of the Hamiltonian

$$H_n^{'}(\lambda_2,\dots,\lambda_n)=\sum_{i=2}^n V(\lambda_i)-\frac{2}{n}\sum_{2\le i<j}\log|\lambda_i-\lambda_j|.$$

Thus, the bound for the second factor follows from (2.26) with ρ_n replaced by $\rho_n^{(2a)}$. Finally, from (2.89) and (2.89) we derive

$$\log\frac{Q^{(1)}_{n,\beta}}{Q_{n,\beta}}=\log\frac{Q^{(1)}_{n,\beta}}{Q^{(1a)}_{n,\beta}}+\log\frac{Q^{(1a)}_{n,\beta}}{Q_{n,\beta}}\le C(n^{1/2}\log n+nd_n).\tag{2.92}$$

The assertion of Theorem 2.3 follows. □

2.3 Auxiliary Results

Proof of Lemma 2.1 Define $F:[0,1]\to\mathbb{R}_+$ as

$$F(t)=T\log\int\exp\{-T^{-1}((1-t)\mathcal{H}_1(\Lambda)-t\mathcal{H}_2(\Lambda))\}d\Lambda.$$

It is evident that $F''(t)\ge 0$. Therefore we have for all $t\in[0,1]$:

$$F'(0)\le F'(t)\le F'(1),$$

and integrating with respect to t, we get

$$F'(0)\le F(1)-F(0)\le F'(1).$$

Inequality (2.38) follows. □

Proof of Theorem 2.1 (i) We prove first that there exists some n-independent C, such that

$$\int\rho_n(\lambda)V(\lambda)d\lambda\le C. \tag{2.93}$$

Choosing in (2.38) $T=2/\beta n$, $H_1=H$ and $H_1=H^{(\epsilon)}$, where $H^{(\epsilon)}$ has the form (2.8) with V replaced by $(1-\epsilon_1)V$, $\epsilon_1=\epsilon/2(1+\epsilon)$, we get from (2.38)

$$\epsilon_1\int\rho_n(\lambda)V(\lambda)d\lambda\le\frac{2}{n^2\beta}\log Q_{n,\beta}^{(\epsilon)}-\frac{2}{n^2\beta}\log Q_{n,\beta}.$$

Now (2.93) follows from the inequalities:

$$\frac{2}{n^2\beta}\log Q_{n,\beta}^{(\epsilon)}\le -m,\qquad \frac{2}{n^2\beta}\log Q_{n,\beta}\ge -M-3/2$$

with M and m defined in (2.25). The first can be easily obtained by the Laplace method, if we use the bound

$$\log|\lambda-\mu|\le\log(1+|\lambda|)+\log(1+|\mu|), \tag{2.94}$$

and the fact that $(1-\epsilon_1)V$ satisfies (1.2). And the second follows from the Jensen inequality:

$$\begin{aligned}Q_{n,\beta}^{(\epsilon)}&>\int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2}d\Lambda\exp\{-H(\Lambda)\}\\&\ge\exp\left\{-\int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2}H(\Lambda)d\Lambda\right\}\ge e^{\beta n^2C_1/2}\end{aligned}$$

with $\Lambda=(\lambda_1,\dots,\lambda_n)$ and

$$C_1=-\int_{-1/2}^{1/2}|V(\lambda)|d\lambda+\int_{-1/2}^{1/2}\log|\lambda-\mu|d\lambda d\mu\ge -M-3/2.$$

Denote for the moment $H(\lambda_1,\dots,\lambda_n)$ of (2.8) as $H_n(\lambda_1,\dots,\lambda_n)$. Then (2.8) implies

$$\frac{\beta n}{2}H_n(\lambda_1,\dots,\lambda_n)=\frac{\beta(n-1)}{2}H_{n-1}(\lambda_2,\dots,\lambda_n)+\frac{\beta n}{2}V(\lambda_1)+\frac{\beta}{2}\sum_{i=2}^{n}(V(\lambda_i)-2\log|\lambda_1-\lambda_i|),\tag{2.95}$$

and in view of (2.94) and (1.2) we obtain

$$\frac{\beta n}{2}H_n(\lambda_1,\dots,\lambda_n)\ge\frac{\beta(n-1)}{2}H_{n-1}(\lambda_2,\dots,\lambda_n)+\frac{\beta n}{2}V(\lambda_1)-\beta(n-1)\log(1+|\lambda_1|).$$

This and (2.9) yield:

$$\begin{aligned}\int Q_{n-1,\beta}^{-1}&\exp\Big\{-\frac{\beta(n-1)}{2}H_{n-1}(\lambda_2,\dots,\lambda_n)\Big\}d\lambda_2\dots d\lambda_n\\&\le\exp\Big\{-\frac{\beta n}{2}V(\lambda_1)+\beta(n-1)\log(1+|\lambda_1|)\Big\}\\&\le\exp\Big\{-\frac{\beta n\epsilon}{2(1+\epsilon)}V(\lambda_1)\Big\}.\end{aligned}\tag{2.96}$$

On the other hand, by using again (2.95) and the Jensen inequality for the "Gibbs" measure $e^{-\beta(n-1)H_{n-1}/2}Q_{n-1,\beta}^{-1}$, we obtain

$$Q_{n-1,\beta}^{-1}Q_{n,\beta}\ge\int_{-1/2}^{1/2}d\lambda_1\exp\Big\{-\frac{\beta n}{2}u_n(\lambda_1;\overline{N}_{n-1})-\frac{\beta(n-1)}{2}\int V(\lambda)\rho_{n-1}(\lambda)d\lambda\Big\},$$

where $\overline{N}_{n-1}$ is defined in (2.5–2.6) and u_n is defined in (2.45).

Using the Jensen inequality with respect to ν_0, the Lebesgue measure on the interval $[-1/2,1/2]$, we get further

$$Q_{n-1,\beta}^{-1}Q_{n,\beta}\ge e^{-(n-1)\beta C/2}\exp\Big\{-\frac{n\beta}{2}\int_{-1/2}^{1/2}V(\lambda)d\lambda-(n-1)\beta\mathcal{L}[\nu_0,\overline{N}_{n-1}]\Big\},\tag{2.97}$$

where C is defined in (2.93). But since

$$-\mathcal{L}(\lambda;\nu_0)=(1/2-\lambda)\log(1/2-\lambda)+(1/2+\lambda)\log(1/2+\lambda)-1\ge-1-\log 2,\tag{2.98}$$

we have

$$-\mathcal{L}[\nu_0,\overline{N}_{n-1}]=-\int\mathcal{L}(\lambda;\nu_0)\overline{N}_{n-1}(d\lambda)\ge-1-\log 2,$$

hence

$$Q_{n-1,\beta}^{-1}Q_{n,\beta}\ge e^{-n\beta C_1/2},\quad C_1=C+2+2\log 2+M,\tag{2.99}$$

and

$$\rho_n(\lambda) = \frac{Q_{n-1,\beta}}{Q_{n,\beta}} Q_{n-1,\beta}^{-1} \int e^{-\beta n H(\lambda_1,\dots,\lambda_n)/2} d\lambda_2 \dots d\lambda_n$$
$$\le e^{\beta n C_1} e^{-\beta n \epsilon V(\lambda)/2(1+\epsilon)} \le e^{-\beta n \epsilon V(\lambda)/4(1+\epsilon)}, \quad |\lambda| > L, \tag{2.100}$$

if L is big enough. This proves the first bound in (2.24). The bound (2.24) for $p_{2,\beta}^{(n)}$ can be proved analogously. □

Proof of Lemma 2.2 (i) Using (2.18), it is easy to see that $\Phi_n(m)$ is convex, i.e.,

$$\Phi_n\left[\frac{m^{(1)} + m^{(2)}}{2}\right] \le \frac{\Phi_n[m^{(1)}] + \Phi_n[m^{(2)}]}{2}. \tag{2.101}$$

Let us show that $\Phi_n[m]$ is bounded from below. Let ν_0 be the Lebesgue measure on the interval $[-1/2, 1/2]$. Then, using the Jensen inequality and then (2.98), we get similarly to (2.97–2.99)

$$\frac{2}{\beta n} \log \int_{-1/2}^{1/2} \exp\left\{-\frac{\beta n}{2} V(\lambda) - \beta(n-1)\mathcal{L}(\lambda, m)\right\} d\lambda$$
$$\ge -\int \mathcal{L}(\lambda; \nu_0) m(d\lambda) - \int_{-1/2}^{1/2} V(\lambda) d\lambda > -(1+\log 2) - \int_{-1/2}^{1/2} V(\lambda) d\lambda. \tag{2.102}$$

Combining this inequality with (2.18) we conclude that $\inf \Phi_n[m] > -\infty$.

Consider a minimizing sequence $\{m^{(k)}\}$ of measures, satisfying (2.43) and such that

$$\lim_{k\to\infty} \Phi_n[m^{(k)}] = \inf_{m\in\mathcal{C}^*} \Phi_n[m] := \Phi_n^*.$$

Then for any $\varepsilon > 0$ there exists k_ε such that

$$\Phi_n^* + \varepsilon \ge \Phi_n[m^{(k)}] \ge \Phi_n^*, \quad k > k_\varepsilon.$$

This and (2.101) yield for $k, l > k_\varepsilon$,

$$\Phi_n^* + \varepsilon \ge \frac{\Phi_n[m^{(k)}] + \Phi_n[m^{(l)})]}{2} \ge \Phi_n\left[\frac{m^{(k)} + m^{(l)}}{2}\right] \ge \Phi_n^*.$$

Besides, it follows from (2.46) that

$$\Phi_n[m] = \frac{n-1}{n} \mathcal{L}[m, m] + \Psi_n[m],$$

where Ψ_n is also convex. Using the convexity of Ψ_n and the previous inequality, we obtain:

$$\mathcal{L}[m^{(k)} - m^{(l)}, m^{(k)} - m^{(l)}]$$
$$\le 4\left(\frac{\Phi_n[m^{(k)}] + \Phi_n[m^{(l)}]}{2} - \Phi_n\left[\frac{m^{(k)} + m^{(l)}}{2}\right]\right) \le 2\varepsilon. \tag{2.103}$$

In other words, the sequence $\{m^{(k)}\}, m^{(k)} \subset \mathcal{C}^*$ of (2.43) satisfies the Cauchy condition with respect to the norm $\|m\|_* = \mathcal{L}^{1/2}[m, m]$ and, as a result, has a limit point m_n in this cone by Proposition 2.1 (v). This point m_n is a minimum point for Φ_n. Besides, since the second derivative of Φ_n in any direction is bounded from below by a positive constant, m_n is a unique minimum point.

Consider the measure $N_n^{(a)}(d\lambda; m_n)$, defined by (2.51) for $m = m_n$. Taking the derivative of $\Phi_n(m_n + t(m - m_n))$ with respect to t at $t = 0$ it is easy to find that for any $m \in \mathcal{C}^*$

$$\mathcal{L}[m_n - N_n^{(a)}, m - m_n] = 0. \tag{2.104}$$

Let us show that for any $m \in \mathcal{C}^*$ we have for $\rho_n^{(a)}$ of (2.49)

$$\rho_n^{(a)}(\lambda; m) \le e^{-nCV(\lambda)}, \quad |\lambda| > 1/2. \tag{2.105}$$

Since $\operatorname{supp} m \subset [-1/2, 1/2]$ and $m(\mathbb{R}) \le 1$, we have

$$-\mathcal{L}(\lambda; m) \le \log(1 + |\lambda|).$$

This and (1.2) yield for the numerator of (2.49)

$$e^{-\beta n u_n(\lambda, m)} \le e^{-\beta n \epsilon V(\lambda)/2(1+\epsilon)}.$$

To estimate the denominator in (2.49) we use (2.102). Then, using the last inequality in (2.100) we get (2.105). We recall here that we use the scaling of the λ-axis such that $L < 1/2$.

Consider now the measure

$$\widetilde{N}_n^{(a)}(\lambda) = N_n^{(a)}(\lambda; m_n)\mathbf{1}_{|\lambda|<1/2}.$$

It follows from (2.105) that

$$|\mathcal{L}(\lambda; \widetilde{N}_n^{(a)}) - \mathcal{L}(\lambda; N_n^{(a)})| \le e^{-nc}/2. \tag{2.106}$$

Thus

$$\mathcal{L}[N_n^{(a)} - \widetilde{N}_n^{(a)}, N_n^{(a)} - \widetilde{N}_n^{(a)}] \le e^{-nc}/2.$$

Besides, replacing in (2.104) m by $\widetilde{N}_n^{(a)}$, we get

$$\mathcal{L}[m_n - N_n^{(a)}, m_n - \widetilde{N}_n^{(a)}] = 0,$$

hence (2.52) follows.

(ii) Define (cf. (2.46))

$$\Phi_n^{(1)}[m] = \frac{(n-1)}{n}\mathcal{L}[m, m] + \frac{2}{\beta n}\log\int_{-1/2}^{1/2} d\lambda e^{-\beta n u_n(\lambda; m)/2}, \tag{2.107}$$

where u_n is given by (2.45). Then (2.105) implies for any Φ_n of (2.46) and $m \in \mathcal{C}^*$ of (2.43)

$$|\Phi_n^{(1)}[m] - \Phi_n[m]| \le e^{-nc}. \tag{2.108}$$

Repeating the proof of existence of a minimizer Φ_n in (i), we obtain that there exists a unique measure $m_n^{(1)} \in \mathcal{C}^*$ such that

$$\Phi_n^{(1)}[m_n^{(1)}] = \inf_{m \in \mathcal{C}^*} \Phi_n^{(1)}[m]. \tag{2.109}$$

Now, if we define (cf. (2.49), (2.51))

$$N_n^{(a,1)}(d\lambda) = e^{-\beta n u_n(\lambda; m_n^{(1)})/2} \mathbf{1}_{|\lambda| \le 1/2} \left(\int_{-1/2}^{1/2} e^{-\beta n u_n(\lambda; m_n^{(1)})/2} d\lambda \right)^{-1}, \tag{2.110}$$

then the analog of (2.104) for $\Phi_n^{(1)}$ implies the equation

$$m_n^{(1)} = N_n^{(a,1)}. \tag{2.111}$$

Consider $F : [0, 1] \to \mathbb{R}$, given by

$$\begin{aligned} F(t) = {} & \frac{(n-1)}{n} \mathcal{L}[m_n^{(1)} + t(N - m_n^{(1)}), m_n^{(1)} + t(N - m_n^{(1)})] \\ & - (1-t) \int u_n(\lambda; m_n^{(1)}) m_n^{(1)}(d\lambda) - t \int u(\lambda; N) N(d\lambda), \end{aligned} \tag{2.112}$$

where u and u_n are defined in (2.13) and (2.45). It is evident, that $F''(t) \ge 0$ and we obtain in view of (2.12–2.13)

$$\begin{aligned} F(1) - F(0) \le F'(1) = {} & 2\frac{n-1}{n} \mathcal{L}[N, N - m_n^{(1)}] \\ & + \int u_n(\lambda; m_n^{(1)}) m_n^{(1)}(d\lambda) - \int V(\lambda) N(d\lambda) - 2\mathcal{L}[N, N] \\ = {} & \int u_n(\lambda; m_n^{(1)}) m_n^{(1)}(d\lambda) - \int u_n(\lambda; m_n^{(1)}) N(d\lambda) + O(n^{-1}). \end{aligned} \tag{2.113}$$

This inequality and (2.108) imply

$$\begin{aligned} 0 & \le \Phi_n[N] - \Phi_n[m_n] \\ & = (\Phi_n^{(1)}[N] - F(1)) + (F(1) - F(0)) + (F(0) - \Phi_n^{(1)}[m_n^{(1)}]) + O(e^{-nc}) \\ & \le (\Phi_n^{(1)}[N] - F(1)) + \int u_n(\lambda; m_n^{(1)}) m_n^{(1)}(d\lambda) - \int u_n(\lambda; m_n^{(1)}) N(d\lambda) \\ & \quad + (F(0) - \Phi_n^{(1)}[m_n^{(1)}]) + O(n^{-1}), \end{aligned} \tag{2.114}$$

where $\Phi_n^{(1)}(m)$ and $m_n^{(1)}$ are defined in (2.107) and (2.109).

Therefore to prove (2.53) it suffices to have the inequalities:

$$\begin{aligned} & \Phi_n^{(1)}[N] - F(1) \le 0; \\ & \int u_n(\lambda; m_n^{(1)}) m_n^{(1)}(d\lambda) - \int u_n(\lambda; m_n^{(1)}) N(d\lambda) \le C n^{-1} \log n; \\ & F(0) - \Phi_n^{(1)}[m_n^{(1)}] \le C n^{-1} \log n. \end{aligned} \tag{2.115}$$

The first inequality follows from (2.105), (2.14–2.15) and the simple bound

$$\frac{2}{\beta n} \log \int_{-1/2}^{1/2} e^{-\beta n u(\lambda; N)/2} d\lambda \le -\min_{\lambda \in \mathbb{R}} u(\lambda; N) = -\int u(\lambda; N) N(d\lambda),$$

where the last equality follows from (2.14).

To prove the second inequality in (2.115) we introduce the function

$$\delta_n(\lambda)=\frac{n^{1/\gamma}}{2}\mathbf{1}_{|\lambda|<n^{-1/\gamma}}, \tag{2.116}$$

and consider the convolution operator δ_n^* defined for any finite measure m as

$$(\delta_n^* m)(\Delta)=\int_\Delta \frac{n^{1/\gamma}}{2}(m(\lambda+n^{-1/\gamma})-m(\lambda-n^{-1/\gamma}))d\lambda, \quad m(\lambda)=m((-\infty,\lambda]).$$

It is evident that for any non-negative measure m such that $m(\mathbb{R})\le 1$ the measure $\delta_n^* m$ has a density bounded by $n^{1/\gamma}$. This implies, in particular, that

$$|\mathcal{L}(\lambda+h;\delta_n^* m)-\mathcal{L}(\lambda;\delta_n^* m)|\le n^{1/\gamma}\int_{-1}^{1}\left|\log\left|1+\frac{h}{\lambda-\mu}\right|\right|d\lambda\le Ch^{1/2}n^{1/\gamma}. \tag{2.117}$$

Besides, if the measure m is absolutely continuous and its density is ρ, then $\delta_n^* m$ has the density

$$(\delta_n*\rho)(\lambda)=\int \delta_n(\lambda-\mu)\rho(\mu)d\mu,$$

the convolution of δ_n and ρ. We will also use below the following estimate valid for any function $v:\mathbb{R}\to\mathbb{C}$, satisfying the Hölder condition with the exponent γ:

$$|(\delta_n*v)(\lambda)-v(\lambda)|\le(2n^{-1/\gamma})^{-1}\int_{|\mu|\le n^{-1/\gamma}}|v(\lambda+\mu)-v(\lambda)|d\mu\le Cn^{-1/\gamma}. \tag{2.118}$$

Moreover, for any m with finite energy (2.12) we have

$$\int \delta_n(\lambda-\mu)\mathcal{L}(\mu;m)d\mu=\mathcal{L}(\lambda;\delta_n^* m), \tag{2.119}$$

and in view of the relations

$$\hat{\delta}_n(p):=\int e^{ip\lambda}\delta_n(\lambda)d\lambda=\frac{\sin pn^{-1/\gamma}}{pn^{-1/\gamma}}, \quad |\hat{\delta}_n(p)|\le 1,$$

and (2.21) we obtain

$$\mathcal{L}[\delta_n^* m-m,m]=\int_0^\infty \frac{|\hat{m}(p)|^2}{p}(\hat{\delta}_n(p)-1)dp\le 0,$$

hence

$$\mathcal{L}[\delta_n^* m,m]\le\mathcal{L}[m,m]. \tag{2.120}$$

Now we are ready to prove the second and the third inequality in (2.115). Using (2.111), (2.38) with $\mathcal{H}_1=u_n(\lambda;m_n^{(1)})$, $\mathcal{H}_2=0$ and $T=2/(\beta n)$, and then (2.108) we have

$$-\int u_n(\lambda;m_n^{(1)})m_n^{(1)}(d\lambda)$$

$$=-\int u_n(\lambda;m_n^{(1)})N_n^{(a,1)}(d\lambda)$$

$$\begin{aligned}&\geq \frac{2}{\beta n}\log\int_{-1/2}^{1/2} e^{-\beta n u_n(\lambda;m_n^{(1)})/2}d\lambda - \frac{2}{\beta n}\log\int_{-1/2}^{1/2} d\lambda\\ &= \frac{2}{\beta n}\log\int e^{-\beta n u_n(\lambda;m_n^{(1)})/2}d\lambda + O(e^{-nc}).\end{aligned} \tag{2.121}$$

Besides, we have by Jensen inequality

$$\begin{aligned}\frac{2}{\beta n}\log\int e^{-\beta n u_n(\lambda;m_n^{(1)})/2}d\lambda &= \frac{2}{\beta n}\log\int \delta_n(\lambda-\mu)e^{-\beta n u_n(\lambda;m_n^{(1)})/2}d\mu d\lambda\\ &\geq \frac{2}{\beta n}\log\int e^{-\beta n \check{u}_n(\mu)/2}d\mu,\end{aligned} \tag{2.122}$$

where

$$\check{u}_n(\lambda) = (\delta_n * V)(\lambda) + 2\frac{n-1}{n}\mathcal{L}(\lambda;\delta_n^* m_n^{(1)}).$$

Observe also that if

$$\check{u}_n^* := \min_{\lambda\in\mathbb{R}} \check{u}_n(\lambda) = \check{u}_n(\lambda^*),$$

then (2.117) with $h = n^{-6/\gamma}$ implies

$$\check{u}_n(\lambda) < \check{u}_n^* + Cn^{-2}, \qquad |\lambda-\lambda^*| \leq n^{-6/\gamma},$$

thus

$$\int d\lambda e^{-\beta n\check{u}_n(\lambda)/2} \geq n^{-6/\gamma} e^{-\beta n \check{u}_n^*/2 - Cn^{-1}}.$$

This bound, (2.121), and (2.122) yield

$$\begin{aligned}&-\int u_n(\lambda;m_n^{(1)})m_n^{(1)}(d\lambda)\\ &\quad\geq \frac{2}{\beta n}\log\int_{-1/2}^{1/2} e^{-\beta n\check{u}_n(\lambda)/2}d\lambda + O(e^{-nc}) \geq -\check{u}_n^* - Cn^{-1}\log n.\end{aligned} \tag{2.123}$$

Using this inequality and (2.118) for $v(\lambda) = \mathcal{L}(\lambda;N)$ and $v(\lambda) = V(\lambda)$, we obtain in view of (2.17), (2.45), and (2.116)

$$\begin{aligned}&-\int u_n(\lambda;m_n^{(1)})m_n^{(1)}(d\lambda)\\ &\geq -\int \check{u}_n(\lambda)N(d\lambda) - Cn^{-1}\log n\\ &= -2\frac{n-1}{n}\int(\delta_n * \mathcal{L}(\,\cdot\,;N))(\lambda)m_n^{(1)}(d\lambda) - \int(\delta_n * V)(\lambda)N(d\lambda) - Cn^{-1}\log n\\ &= -2\frac{n-1}{n}\mathcal{L}[N,m_n^{(1)}] - \int V(\lambda)N(d\lambda) + O(n^{-1}) - Cn^{-1}\log n\\ &= -\int u_n(\lambda;m_n^{(1)})N(d\lambda) + O(n^{-1}\log n).\end{aligned}$$

Hence, we have proved the second inequality in (2.115).

By a similar argument we derive from (2.122), (2.123), (2.118) and (2.120) that

$$\begin{aligned}
&\frac{2}{\beta n}\log\int_{-1/2}^{1/2} e^{-\beta n u_n(\lambda;m_n^{(1)})/2}d\lambda\\
&\quad\ge -2\frac{n-1}{n}\mathcal{L}[\delta_n^* m_n^{(1)}, m_n^{(1)}] - \int(\delta_n * V)(\lambda)m_n^{(1)}(d\lambda) - Cn^{-1}\log n\\
&\quad\ge -2\frac{n-1}{n}\mathcal{L}[m_n^{(1)}, m_n^{(1)}] - \int V(\lambda)m_n^{(1)}(d\lambda) + O(n^{-1}) - Cn^{-1}\log n\\
&\quad\ge -\int u_n(\lambda; m_n^{(1)})m_n^{(1)}(d\lambda) + O(n^{-1}\log n).
\end{aligned}$$

In view of (2.107) and (2.112)

$$F(0) - \Phi^{(1)}[m_n^{(1)}] = -\int u_n(\lambda; m_n^{(1)})m_n^{(1)}(d\lambda) - \frac{2}{\beta n}\int_{-1/2}^{1/2} e^{-\beta n u_n(\lambda;m_n^{(1)})/2}d\lambda$$

and the third inequality of (2.115) follows. Lemma 2.2 is proved. □

3 Bulk Universality of Local Eigenvalue Statistics

3.1 Generalities

Universality is an important asymptotic property of spectra of random matrices of large size n. According to the property (see e.g. [10, 13, 16]) the probabilistic description of eigenvalues on the scale of typical spacing does not depend on the matrix probability law (ensemble) in the limit $n \to \infty$ and may only depend on the type of matrices (real symmetric, hermitian, or quaternion real in the case of real eigenvalues and orthogonal, unitary or symplectic in the case of the eigenvalues on the unit circle).

In a more concrete setting of the bulk of the spectrum of hermitian matrix models (1.1–1.3) the property can be described as follows. Assume that the limiting Normalized Counting Measure of eigenvalues N (see e.g. Theorem 2.1 for its existence) possesses a continuous density ρ (see e.g. Theorem 2.2). Choose λ_0 belonging to the bulk of the support of N, i.e., such that $0 < \rho(\lambda_0) < \infty$, and assume that ρ_n of (2.6) converges uniformly to ρ in a neighborhood of λ_0. Then we have to have the following limiting relation for any marginal density (2.4) for $\beta = 2$:

$$\begin{aligned}
&\lim_{n\to\infty}[\rho_n(\lambda_0)]^{-l} p_{l,2}^{(n)}\left(\lambda_0 + \frac{x_1}{n\rho_n(\lambda_0)}, \dots, \lambda_0 + \frac{x_l}{n\rho_n(\lambda_0)}\right)\\
&\quad = \det\{S(x_1 - x_k)\}_{j,k=1}^{l}, \qquad (3.1)
\end{aligned}$$

where

$$S(x) = \frac{\sin \pi x}{\pi x}. \qquad (3.2)$$

In other words, the limit in the r.h.s. of (3.1) should not depend on V in (1.1) (modulo some weak conditions) for all λ_0 that belong to the bulk of the spectrum. Note that the r.h.s. of (3.1) does not depend on λ_0.

Thus the limit (3.1) for arbitrary V has to coincide with that for the archetype Gaussian Unitary Ensemble, corresponding to $V(\lambda)=\lambda^2/2$. For this case (3.1) is known since the early sixties (see [16] for corresponding results and discussions).

In addition, an analogous properties has to be valid for the "hole" probability

$$E_{n,2}(\Delta)=\mathbf{P}\{\lambda_l^{(n)}\notin\Delta, l=1,\dots,n\}, \quad \Delta\subset\mathbb{R}. \tag{3.3}$$

Namely, we have to have for any $s>0$:

$$\lim_{n\to\infty}E_{n,2}([\lambda_0,\lambda_0+s/n\rho_n(\lambda_0)])=\det(1-S_s) \tag{3.4}$$

where S_s is the integral operator, defined by the kernel $S(x-y)$ on the interval $[0,s]$.

We will prove the following

Theorem 3.1 *Consider a matrix model* (1.1–1.3) *for* $\beta=2$ *and assume that its potential* V *satisfies* (1.2), V' *is a Lipschitz function* (*see* (2.29)) *and there exists a closed interval* $[a,b]\subset\sigma=\mathrm{supp}N$ *such that*

$$\sup_{\lambda\in[a,b]}|V'''(\lambda)|\le C_1<\infty, \qquad 0<\inf_{\lambda\in[a,b]}\rho(\lambda). \tag{3.5}$$

Then for any $d>0$ *the universality properties* (3.1) *and* (3.4) *are true for any* $\lambda_0\in[a+d,b-d)$. *More precisely*

(i) (3.1) *is true uniformly in* $(x_1,\dots,x_l)$, *varying on a compact set of* $\mathbb{R}^l$;
(ii) (3.4) *is true uniformly in* s, *varying on a compact set of* $[0,\infty)$.

The theorem will be proved in this and the next subsections. An important technical mean of the proof is a remarkable formula for all marginals (2.4) of the joint eigenvalue probability density (2.2) for $\beta=2$. The formula is known as the determinant formula (see e.g. [16] for details).

Assume that V satisfies (1.2) and consider polynomials $\{P_l^{(n)}(\lambda)\}_{l\ge0}$ orthogonal on $\mathbb{R}$ with respect to the weight

$$w_n(\lambda)=e^{-nV(\lambda)}. \tag{3.6}$$

We have

$$\int P_l^{(n)}(\lambda)P_m^{(n)}(\lambda)e^{-nV(\lambda)}d\lambda=\delta_{l,m}, \tag{3.7}$$

or, denoting

$$\psi_l^{(n)}(\lambda)=\exp\{-nV(\lambda)/2\}P_l^{(n)}(\lambda), \quad l=0,1,\dots, \tag{3.8}$$

we obtain the corresponding orthogonal functions in $L^2(\mathbb{R})$:

$$\int\psi_l^{(n)}(\lambda)\psi_m^{(n)}(\lambda)d\lambda=\delta_{l,m}. \tag{3.9}$$

Then marginal densities (2.4) have the determinant form [16]

$$p_{l,2}^{(n)}(\lambda_1,\dots,\lambda_l)=\frac{(n-l)!}{n!}\det\{K_n(\lambda_j,\lambda_k)\}_{j,k=1}^l, \tag{3.10}$$

where

$$K_n(\lambda,\mu)=\sum_{l=0}^{n-1}\psi_l^{(n)}(\lambda)\psi_l^{(n)}(\mu),$$
$$\int K_n(\lambda,\nu)K_n(\nu,\mu)d\mu=K_n(\lambda,\mu) \tag{3.11}$$

is known as the reproducing kernel of system (3.8). In particular,

$$\rho_n(\lambda):=p_{1,2}^{(n)}(\lambda)=n^{-1}K_n(\lambda,\lambda)=n^{-1}\sum_{l=0}^{n-1}(\psi_l^{(n)}(\lambda))^2. \tag{3.12}$$

We mention also the Christoffel–Darboux formula [24]:

$$K_n(\lambda,\mu)=J_{n-1}^{(n)}\frac{\psi_n^{(n)}(\lambda)\psi_{n-1}^{(n)}(\mu)-\psi_{n-1}^{(n)}(\lambda)\psi_n^{(n)}(\mu)}{\lambda-\mu}, \tag{3.13}$$

where

$$J_k^{(n)}=\int\lambda\psi_k^{(n)}(\lambda)\psi_{k+1}^{(n)}(\lambda)d\lambda,\quad k=0,1,\dots \tag{3.14}$$

are the off-diagonal coefficients of the Jacobi matrix, associated with these orthogonal polynomials.

Write the hole probability as

$$E_{n,\beta}(\Delta)=\mathbf{E}\left\{\prod_{l=1}^{n}(1-\chi_\Delta(\lambda_l^{(n)}))\right\},$$

where χ_Δ is the indicator of $\Delta\subset\mathbb{R}$, use the symmetry of (2.2) in its arguments, (3.10), and the scaling of the l.h.s. of (3.4). This yields for $\Delta=[\lambda_0,\lambda_0+s/n\rho(\lambda_0)]$:

$$\begin{aligned}&E_{n,2}([\lambda_0,\lambda_0+s/n\rho_n(\lambda_0)])\\&\quad=\det(I-K_n\chi_\Delta)\\&\quad=1+\sum_{l=1}^{n}\frac{(-1)^l}{l!}\rho_n^{-l}(\lambda_0)\int_{[0,s]}dx_1\dots dx_l\\&\qquad\times\det\left\{K_n\left(\lambda_0+\frac{x_i}{n\rho_n(\lambda_0)},\lambda_0+\frac{x_i}{n\rho_n(\lambda_0)}\right)\right\}_{i,j=1}^{l}\end{aligned} \tag{3.15}$$

where K_n is the integral operator with the kernel K_n of (3.11) and χ_Δ is the multiplication operator by χ_Δ. In view of (3.10) and (3.15) the proof of the universality properties (3.1) and (3.4) for the random matrix ensemble (1.1–1.3) with $\beta=2$ reduces in essence to the proof of the limiting relation

$$\lim_{n\to\infty}(n\rho_n(\lambda_0))^{-1}K_n(\lambda_0+x/n\rho_n(\lambda_0),\lambda_0+y/n\rho_n(\lambda_0))=\frac{\sin\pi(x-y)}{\pi(x-y)}. \tag{3.16}$$

In paper [6] the asymptotic formulas for $\psi_n^{(n)}$, $\psi_{n-1}^{(n)}$, $J_{n-1}^{(n)}$ as $n\to\infty$ were found in the case of a real analytic potential, and the limits (3.1) and (3.4) were obtained by using above formulas, (3.13) in particular. In paper [21] a certain integral representation for $K_n(\lambda,\mu)$ was used (see formula (3.31) below) to obtain the *sin*-kernel of the r.h.s. of (3.16) as a series in its argument. In this paper we start from the same representation of the reproducing kernel and derive an integro-differential equation for the limit of the l.h.s. of (3.16). We then show that a unique solution of the equation is the *sin*-kernel of the r.h.s. of (3.16). It turns out that this requires weaker conditions (see Theorem 3.1) than the potential to be a real analytic function. In view of this and the importance of the universality properties (3.1) and (3.4) it seems reasonable to present one more proof of the property.

3.2 Proof of Basic Results

An important ingredient of our proof is the uniform convergence of ρ_n of (2.6) to ρ of (2.32) in a neighborhood of λ_0.

Theorem 3.2 *Under conditions of Theorem 3.1 we have for any $d>0$:*

$$\sup_{\lambda\in[a+d,b-d]}|\rho_n(\lambda)-\rho(\lambda)|\le Cn^{-2/9} \tag{3.17}$$

with some positive and finite C.

Proof We note first again that we can assume without loss of generality that V is linear for large absolute values of its argument, i.e., that (2.78) is valid (see the beginning of the proof of Theorem 2.2). Using in (2.83) representation (3.10) for $p_2^{(n)}(\lambda,\mu)$, we obtain for $z=\lambda+i\eta$, $\eta>0$:

$$f_n^2(z)+V'(\lambda)f_n(z)+Q_n(\lambda,\eta)=-\frac{1}{n^2}\int K_n^2(\lambda,\mu)\left(\frac{1}{\lambda-z}-\frac{1}{\mu-z}\right)^2 d\lambda d\mu, \tag{3.18}$$

where $f_n(z)$ was defined in (2.82), and

$$Q_n(\lambda,\eta)=\int\frac{V'(\mu)-V'(\lambda)}{\mu-z}\,\rho_n(\mu)d\mu,$$

is well defined due to (3.22), (2.78), and our conditions on $V(\lambda)$ (see Theorem 3.1).

To proceed further we need two lemmas, whose proof will be given in the next subsection.

Lemma 3.1 *Let $K_n(\lambda,\mu)$ be defined by (3.11). Then for any $\delta>0$ we have under conditions of Theorem 2.1:*

$$\int(\lambda-\mu)^2K_n^2(\lambda,\mu)d\lambda d\mu\le C,\qquad \int_{|\lambda-\mu|>\delta}K_n^2(\lambda,\mu)d\lambda d\mu\le C\delta^{-2}, \tag{3.19}$$

$$\left|\int(\lambda-\mu)^\alpha K_n^2(\lambda,\mu)d\mu\right|\le C([\psi_{n-1}^{(n)}(\lambda)]^2+[\psi_n^{(n)}(\lambda)]^2),\quad \alpha=1,2, \tag{3.20}$$

$$\int_{|\lambda-\mu|>\delta}K_n^2(\lambda,\mu)d\mu\le C\delta^{-2}([\psi_{n-1}^{(n)}(\lambda)]^2+[\psi_n^{(n)}(\lambda)]^2). \tag{3.21}$$

Lemma 3.2 *Under the conditions of Theorem 3.1 we have uniformly in $\lambda\in[a+d,b-d]$ for any $d>0$*

$$\rho_n(\lambda)\le C, \tag{3.22}$$

$$\left|\frac{d\rho_n(\lambda)}{d\lambda}\right| \le C([\psi_{n-1}^{(n)}(\lambda)]^2 + [\psi_n^{(n)}(\lambda)]^2) + C, \tag{3.23}$$

$$\int_{|\mu-\lambda|\le n^{-1/4}} d\mu([\psi_{n-1}^{(n)}(\mu)]^2 + [\psi_n^{(n)}(\mu)]^2) \le Cn^{-1/4}, \tag{3.24}$$

$$\frac{1}{n}([\psi_{n-1}^{(n)}(\lambda)]^2 + [\psi_n^{(n)}(\lambda)]^2) \le Cn^{-1/8}. \tag{3.25}$$

It follows then from (3.18) and (3.19) that

$$f_n^2(z) + V'(\lambda) f_n(z) + Q_n(\lambda, \eta) = O(n^{-2}\eta^{-4}), \tag{3.26}$$

where $\eta = |\Im z|$. Observe now that if $\lambda \in [a+d, b-d]$, then we have for sufficiently small η in view of (2.29), (2.78), and (3.22):

$$\begin{aligned}
&|Q_n(\lambda,\eta) - Q_n(\lambda,0)| \\
&\quad \le \eta \int_{|\mu-\lambda|>d/2} \frac{|V'(\mu) - V'(\lambda)|\rho_n(\mu)d\mu}{|\mu-\lambda\|(\mu-\lambda)^2+\eta^2|^{1/2}} + C\eta \int_{|\mu-\lambda|<d/2} \frac{d\mu}{|(\mu-\lambda)^2+\eta^2|^{1/2}} \\
&\quad \le C\eta d^{-2} + C\eta \log \eta^{-1}.
\end{aligned}$$

Besides, applying (2.26), we get

$$Q_n(\lambda, 0) = Q(\lambda) + O(n^{-1/2}\log^{1/2} n),$$

where Q is defined by (2.33). The last two bounds yield

$$Q_n(\lambda,\eta) = Q(\lambda) + O(n^{-1/2}\log^{1/2} n) + O(\eta \log \eta^{-1}). \tag{3.27}$$

Combining (3.26) and (3.27), we find for any $\eta \ge n^{-3/8}$ that

$$\begin{aligned}
f_n(\lambda + i\eta) = -\frac{1}{2}V'(\lambda) + [V'^2(\lambda)/4 - Q(\lambda) + O(\eta\log\eta^{-1}) \\
+ O(n^{-1/2}\log^{1/2} n) + O(n^{-2}\eta^{-4})]^{1/2}.
\end{aligned} \tag{3.28}$$

This and (2.32) yield for $\eta = n^{-4/9}$:

$$\pi^{-1}\Im f_n(\lambda + i\eta) = \rho(\lambda) + O(n^{-2/9}). \tag{3.29}$$

On the other hand, integrating by parts and using (3.28) and Lemma 3.2, we obtain for $\eta = n^{-4/9}$

$$\begin{aligned}
&|\pi^{-1}\Im f_n(\lambda+i\eta) - \rho_n(\lambda)| \\
&\quad \le \frac{\eta}{\pi}\left(\int_{|\mu-\lambda|<\eta^{1/2}} + \int_{\eta^{1/2}<|\mu-\lambda|<d/2}\right)\frac{|\rho_n(\mu)-\rho_n(\lambda)|}{(\mu-\lambda)^2+\eta^2} d\mu + O(\eta) \\
&\quad \le C\int_{|\mu-\lambda|<\eta^{1/2}} |\rho_n'(\mu)| d\mu + O(\eta^{1/2}) \le C\eta^{1/2}.
\end{aligned}$$

This bound and (3.29) imply (3.17). $\square$

Proof of Theorem 3.1 According to (3.10), the proof of validity of (3.1–3.2) uniformly on a compact set of $\mathbb{R}^l$, i.e., assertion (i) of the theorem, reduces to the proof of validity of limiting relation (3.16) for the reproducing kernel (3.11) of the orthonormal systems (3.8) uniformly in (x, y) on a compact set of $\mathbb{R}^2$. This proof occupies the overwhelming part of the this and the next subsections. Before presenting the proof we will show that (3.16) implies (3.4), i.e., assertion (ii) of the theorem. Indeed, if (3.16) is valid, then we can pass to the limit $n \to \infty$ in the integrals over $(x_1, \dots, x_l)$ in every term of (3.15) and obtain the r.h.s. of (3.1) as the integrand of every integral. We have to prove then that the terms of (3.15) are bounded uniformly in n by terms of a convergent series. This is based on

Lemma 3.3 *Let $A = \{A_{jk}\}_{j,k=1}^l$ be a positive definite $l \times l$ matrix. Then*

$$\det A \le \prod_{j=1}^{l} A_{jj}. \tag{3.30}$$

The lemma will be proved in the next subsection. It follows from (3.11) that the matrix $\{K_n(\lambda_j, \lambda_k)\}_{j,k=1}^l$ is positive definite. Hence we have by the above lemma, (3.12), and (3.22):

$$\det\{(n\rho_n(\lambda_0))^{-1} K_n(\lambda_0 + x_j/n\rho_n(\lambda_0), \lambda_0 + x_k/n\rho_n(\lambda_0))\}_{j,k=1}^l$$
$$\le \prod_{j=1}^{l} \frac{\rho_n(\lambda_0 + x_j/n\rho_n(\lambda_0))}{\rho_n(\lambda_0)} \le C^l.$$

Thus, the lth term of (3.15) is bounded by $C^l/l!$, the term of a convergent series. This allows us to pass to the limit $n \to \infty$ in every term of (3.15) and to obtain (3.4) in view of (3.12).

We turn now to the proof of validity of (3.16) uniformly in (x, y) on a compact set of $\mathbb{R}^2$. This will be based on the representation

$$\begin{aligned} n^{-1}K_n(\lambda, \mu) &= Q_{n,2}^{-1} e^{-n(V(\lambda)+V(\mu))/2} \\ &\quad \times \int \prod_{j=2}^{n} d\lambda_j e^{-nV(\lambda_j)} (\lambda - \lambda_j)(\mu - \lambda_j) \prod_{2 \le j < k \le n} (\lambda_j - \lambda_k)^2 \end{aligned} \tag{3.31}$$

which can be derived from the well-known identities of random matrix theory [16]

$$\prod_{1 \le j < k \le n} (\lambda_j - \lambda_k) = \left(\prod_{l=0}^{n-1} \gamma_l^{(n)} \right)^{-1} \det\{P_{j-1}^{(n)}(\lambda_k)\}_{j,k=0}^{n-1},$$
$$Q_{n,2} = n! \prod_{l=1}^{n} (\gamma_l^{(n)})^{-2},$$

where $\gamma_l^{(n)}$ is the coefficient in front of λ^l in the polynomial $P_l^{(n)}$. Using the first identity with $\lambda_1 = \lambda$ and $\lambda_1 = \mu$ in the r.h.s. of (3.31), integrating the result with respect to $\lambda_2, \dots, \lambda_n$ we obtain the l.h.s. of (3.2), in view of the orthonormality of functions (3.8).

We note again that we can assume without loss of generality that the potential is linear for $|\lambda| > L$, where L is defined in Theorem 2.1, i.e., that (2.78) is valid. Corresponding argument is a version of that leading to (2.42) and (2.78). Indeed, if V_1 and V_2 satisfy the conditions of Proposition 2.2, then $V(\lambda,t)=tV_1(\lambda)+(1-t)V_2(\lambda), t\in[0,1]$ also does. Denote $\overline{N}_n(\cdot,t)$ the measure (2.6) and $\rho_n(\cdot,t)$ its density. Then, by using formulas (3.6–3.12) and (3.31), we obtain for the kernel $K_n(\lambda,\mu,t)$, corresponding to $V(\lambda,t)$:

$$\begin{aligned}\frac{\partial}{\partial t}K_n(\lambda,\mu,t)&=-\frac{n}{2}(\delta V(\lambda)+\delta V(\mu))K_n(\lambda,\mu,t)\\&\quad+\int\delta V(\nu)K_n(\lambda,\nu,t)K_n(\mu,\nu,t)d\nu,\end{aligned}\tag{3.32}$$

where $\delta V=V_1-V_2$. Now, if $V_1(\lambda)=V_2(\lambda)$, $|\lambda|\le L$, then in view of the inequality (see (3.11) and (3.12))

$$K_n^2(\lambda,\mu)\le K_n(\lambda,\lambda)K_n(\mu,\mu)=n^2\rho_n(\lambda)\rho_n(\mu)\tag{3.33}$$

we obtain for $\lambda,\mu\in[a+d,b-d]\subset(a,b)$ $(d>0)$:

$$\left|\frac{\partial}{\partial t}K_n(\lambda,\mu,t)\right|\le n^3\rho_n^{1/2}(\lambda,t)\rho_n^{1/2}(\mu,t)\int_{|\lambda|>L}|\delta V(\nu)|\rho_n(\nu,t)d\nu,$$

It follows from (3.22), (2.24), and (1.2) that the r.h.s. of this inequality is $O(e^{-nc})$ uniformly in $t\in[0,1]$ (cf. (2.41)). Hence, the limit (3.16) for a given potential, satisfying the condition of the theorem, is the same as that for the potential, coinciding with the given for $|\lambda|\le L$ and linear for, say $|\lambda|\ge L+1$.

Now take some $\lambda_0\in[a+d,b-d]$, where $[a,b]$ is defined in (3.5), and denote

$$\mathcal{K}_n(x,y)=n^{-1}K_n(\lambda_0+x/n,\lambda_0+y/n).\tag{3.34}$$

We have from (3.11–3.12), (3.22), and (3.33):

$$\int\mathcal{K}_n(x,z)\mathcal{K}_n(z,y)dz=\mathcal{K}_n(x,y),\qquad \mathcal{K}_n^2(x,y)\le\mathcal{K}_n(x,x)\mathcal{K}_n(y,y),\tag{3.35}$$

$$\begin{aligned}&\mathcal{K}_n(x,x)=\rho_n(\lambda_0+x/n)\le C<\infty,\\&|\mathcal{K}_n(x,y)|\le C<\infty,\quad x,y=o(n).\end{aligned}\tag{3.36}$$

Then, differentiating (3.31) with respect to x, we get (cf. (3.32))

$$\begin{aligned}\frac{\partial}{\partial x}\mathcal{K}_n(x,y)&=-\frac{1}{2}V'(\lambda_0+x/n)\mathcal{K}_n(x,y)\\&\quad+\int\frac{\mathcal{K}_n(x',x')\mathcal{K}_n(x,y)-\mathcal{K}_n(x,x')\mathcal{K}_n(x',y)}{x-x'}dx'.\end{aligned}\tag{3.37}$$

We have the following lemma that will be proved in the next subsection.

Lemma 3.4 *Denote*

$$D(\lambda)=\frac{V'(\lambda)}{2}-\frac{1}{n}\int\frac{K_n(\mu,\mu)d\mu}{\lambda-\mu}.$$

Then under conditions of Theorem 3.1 we have uniformly in any $[a+d,b-d]\subset(a,b)$:

$$|D(\lambda)|\le Cn^{-1/4}\log n.\tag{3.38}$$

The lemma yields

$$\frac{1}{2}V'(\lambda_0+x/n)\mathcal{K}_n(x,y)-\int\frac{\mathcal{K}_n(x',x')\mathcal{K}_n(x,y)}{x-x'}dx'$$
$$=D(\lambda_0+x/n)\mathcal{K}_n(x,y)=O(n^{-1/4}\log n).$$

This allows us to rewrite (3.37) as

$$\frac{\partial}{\partial x}\mathcal{K}_n(x,y)=-\int\frac{\mathcal{K}_n(x,x')\mathcal{K}_n(x',y)}{x-x'}dx'+O(n^{-1/4}\log n). \tag{3.39}$$

Denote

$$\mathcal{L}=\log n. \tag{3.40}$$

For $|x|,|y|\le\mathcal{L}$ we can restrict integration in (3.39) by the domain $|x'|\le 2\mathcal{L}$, replacing $O(n^{-1/4}\log n)$ by $O(\mathcal{L}^{-1})$, where $\mathcal{L}$ is defined by (3.40). This follows from the bound

$$\left|\int_{|x'|>2\mathcal{L}}\frac{\mathcal{K}_n(x,x')\mathcal{K}_n(x',y)}{x-x'}dx'\right|$$
$$\le\mathcal{L}^{-1}\left(\int\mathcal{K}_n^2(x,x')dx'\int\mathcal{K}_n^2(y,x')dx'\right)^{1/2}\le C\mathcal{L}^{-1}, \tag{3.41}$$

and (3.35–3.36).

We will use now the following assertion that will be proved in the next subsection.

Lemma 3.5 *Under conditions of Theorem 3.1 we have uniformly in $|x|,|y|<\mathcal{L}$, $\lambda_0\in[a+d,b-d]$*

$$\left|\frac{\partial}{\partial x}\mathcal{K}_n(x,y)+\frac{\partial}{\partial y}\mathcal{K}_n(x,y)\right|\le C(n^{-1/8}+|x-y|n^{-2}), \tag{3.42}$$

$$|\mathcal{K}_n(x,y)-\mathcal{K}_n(0,y-x)|\le C|x|(n^{-1/8}+|x-y|n^{-2}), \tag{3.43}$$

$$\left|\frac{\partial}{\partial x}\mathcal{K}_n(x,y)\right|\le C,\qquad\int_{|x|\le\mathcal{L}}dx\left|\frac{\partial}{\partial x}\mathcal{K}_n(x,y)\right|^2\le C. \tag{3.44}$$

Denote

$$\mathcal{K}_n^*(x)=\mathcal{K}_n(x,0)\mathbf{1}_{|x|\le\mathcal{L}}+\mathcal{K}_n(\mathcal{L},0)(1+\mathcal{L}-x)\mathbf{1}_{\mathcal{L}<x\le\mathcal{L}+1}$$
$$+\mathcal{K}_n(-\mathcal{L},0)(1+\mathcal{L}+x)\mathbf{1}_{-\mathcal{L}-1\le x<-\mathcal{L}}, \tag{3.45}$$

and observe that if we set $x=0$ in (3.39) and take $|y|\le\mathcal{L}/3$, then similarly to (3.41) we can restrict integration to $|x'|\le 2\mathcal{L}/3$ in the obtained relation, adding $O(\mathcal{L}^{-1})$. This and Lemma 3.5 lead to the equation

$$\frac{\partial}{\partial y}\mathcal{K}_n^*(y)=\int_{|x'|\le 2\mathcal{L}/3}\frac{\mathcal{K}_n^*(x')\mathcal{K}_n^*(y-x')}{x'}dx'+r_n(y)+O(\mathcal{L}^{-1}), \tag{3.46}$$

where

$$r_n(y)=\int_{|x'|\le 2\mathcal{L}/3}\frac{\mathcal{K}_n(0,x')(\mathcal{K}_n(x',y)-\mathcal{K}_n(0,y-x'))}{x'}dx',$$

and assuming that $|y|\le\mathcal{L}/3$ we have by Lemma 3.5

$$r_n(y)=O(n^{-1/8}\log n).$$

Now, using the bound similar to (3.41), we can replace in (3.46) the integral over $|x'|\le 2\mathcal{L}/3$ by the integral over the whole real line. Besides, on the basis of Lemma 3.5 and (3.35–3.36), we obtain

$$\int|\mathcal{K}_n^*(x)|^2dx\le\int|\mathcal{K}_n(x,0)|^2dx+C'\le C,\qquad \int\left|\frac{d}{dx}\mathcal{K}_n^*(x)\right|^2dx\le C. \tag{3.47}$$

Consider the Fourier transform

$$\hat{\mathcal{K}}_n^*(p)=\int\mathcal{K}_n^*(x)e^{ipx}dx,$$

where the integral is defined in the $L^2(\mathbb{R})$ sense, and write $\mathcal{K}_n^*(x)$ as

$$\mathcal{K}_n^*(x)=(2\pi)^{-1}\int\hat{\mathcal{K}}_n^*(p)e^{-ipy}dp. \tag{3.48}$$

Then we have from (3.12) and (3.17):

$$\int\hat{\mathcal{K}}_n^*(p)dp=2\pi\rho(\lambda_0)+o(1), \tag{3.49}$$

and from (3.47) and the Parseval equation:

$$\int p^2|\hat{\mathcal{K}}_n^*(p)|^2dp\le C. \tag{3.50}$$

It follows from (3.11) and (3.34) that the kernel $\mathcal{K}_n(x,y)$ is positive definite:

$$\int_{-\mathcal{L}}^{\mathcal{L}}\mathcal{K}_n(x,y)f(x)\overline{f}(y)dxdy\ge 0,\qquad f\in L_2(\mathbb{R}),$$

and by (3.43) we have for any $f\in L_2(\mathbb{R})$:

$$\int\hat{\mathcal{K}}_n^*(p)|\hat{f}(p)|^2dp\ge -C\|f\|^2_{L^2(\mathbb{R})}(n^{-1/8}\log^4 n+O(\mathcal{L}^{-1})). \tag{3.51}$$

Furthermore, the Parseval equation and (3.43) yield

$$\int|\hat{\mathcal{K}}_n^*(p)-\hat{\mathcal{K}}_n^*(-p)|^2dp=2\pi\int|\mathcal{K}_n^*(x)-\mathcal{K}_n^*(-x)|^2dx\le Cn^{-1/8}\log^3 n. \tag{3.52}$$

We write now by definition of the singular integral

$$\int\frac{\mathcal{K}_n^*(x')\mathcal{K}_n^*(y-x')}{x'}dx'=\lim_{\varepsilon\to+0}\int dx'\mathcal{K}_n^*(x')\mathcal{K}_n^*(y-x')\Re(x'+i\varepsilon)^{-1}. \tag{3.53}$$

In view of the formula

$$\int e^{ipx}\Re(x'+i\varepsilon)^{-1}dx=\pi i e^{-\varepsilon|p|}\mathrm{sgn}p$$

and the Parseval equation we can write the r.h.s. of (3.53) as

$$\begin{aligned}&\frac{1}{2\pi}\lim_{\varepsilon\to+0}\int dpdp'\hat{\mathcal{K}}_n^*(p)\hat{\mathcal{K}}_n^*(p')e^{-ipy}\mathrm{sign}(p-p')e^{-\varepsilon|p-p'|}\\&\quad=\frac{i}{2\pi}\int dpe^{-ipy}\hat{\mathcal{K}}_n^*(p)\int_0^p\hat{\mathcal{K}}_n^*(p')dp'\\&\qquad-\frac{i}{2\pi}\int dpe^{-ipy}\hat{\mathcal{K}}_n^*(p)\int_0^\infty(\hat{\mathcal{K}}_n^*(p')-\hat{\mathcal{K}}_n^*(-p'))dp'.\end{aligned}\tag{3.54}$$

Note that the both integrals are absolutely convergent because $\hat{\mathcal{K}}_n^*\in L^1(\mathbb{R})$ by (3.50). Since the Schwarz inequality and (3.50) imply the bound

$$\begin{aligned}&\left|\int_0^\infty(\hat{\mathcal{K}}_n^*(p')-\hat{\mathcal{K}}_n^*(-p'))dp'\right|\\&\quad\le\left|\int_0^{\mathcal{L}^2}(\hat{\mathcal{K}}_n^*(p')-\hat{\mathcal{K}}_n^*(-p'))dp'\right|+\int_{|p|>\mathcal{L}^2}|\hat{\mathcal{K}}_n^*(p')|dp'\\&\quad\le\mathcal{L}\left(\int|\hat{\mathcal{K}}_n^*(p')-\hat{\mathcal{K}}_n^*(-p')|^2dp'\right)^{1/2}+C\mathcal{L}^{-1},\end{aligned}$$

we get from (3.52–3.54) uniformly in $|y|<\mathcal{L}/3$

$$\int\frac{\mathcal{K}_n^*(x')\mathcal{K}_n^*(y-x')}{x'}dx'=\frac{i}{2\pi}\int dp\hat{\mathcal{K}}_n^*(p)e^{-ipy}\int_0^p\hat{\mathcal{K}}_n^*(p')dp'+O(\mathcal{L}^{-1}).$$

This allows us to transform (3.46) into the following asymptotic relation, valid for $|y|\le\mathcal{L}/3$:

$$\int\hat{\mathcal{K}}_n^*(p)\left(\int_0^p\hat{\mathcal{K}}_n^*(p')dp'-p\right)e^{-ipy}dp=O(\mathcal{L}^{-1}).\tag{3.55}$$

Now consider the functions

$$F_n(p)=\int_0^p\hat{\mathcal{K}}_n^*(p')dp'.\tag{3.56}$$

Since $\hat{\mathcal{K}}_n^*(p)\in L^2(\mathbb{R})$, the sequence $\{F_n(p)\}$ consists of functions that are of uniformly bounded variation, uniformly bounded and equicontinuous on $\mathbb{R}$. Thus $\{F_n(p)\}$ is a compact family with respect to the uniform convergence. Hence, the limit F of any subsequence $\{F_{n_k}\}$ possesses the properties:

(a) F is bounded and continuous;
(b) $F(p)=-F(-p)$ (see (3.51));
(c) $F(p)\le F(p')$, if $p\le p'$ (see (3.51));
(d) $F(+\infty)-F(-\infty)=2\pi\rho(\lambda_0)$ (see (3.49));

(e) F satisfies the following equation, valid for any smooth function g of compact support (see (3.55)):

$$\int (F(p) - p)g(p)dF(p) = 0. \tag{3.57}$$

The last property implies that $F(p) = p$ or $F(p) = \text{const}$, hence it follows from (a)–(c) that

$$F(p) = p\mathbf{1}_{|p|\le p_0} + \rho^*\text{sign}(p)\mathbf{1}_{|p|\ge p_0},$$

where $p_0 = \pi\rho(\lambda_0)$ by (d).

We conclude that (3.57) is uniquely soluble, thus the sequence $\{F_n\}$ converges uniformly on any compact to the above F. This and (3.56) imply the weak convergence of the sequence $\{\mathcal{K}_n^*\}$ to the function

$$\mathcal{K}^*(x) = \frac{\sin(\pi\rho(\lambda_0)x)}{\pi\rho(\lambda_0)x}.$$

But weak convergence combined with (3.36) and (3.44) implies the uniform convergence of $\{\mathcal{K}_n^*\}$ to $\mathcal{K}^*$ on any interval. Now, using Lemma 3.5, we obtain that we have uniformly in (x, y), varying on a compact set of $\mathbb{R}^2$

$$\lim_{n\to\infty} \mathcal{K}_n(x, y) = \mathcal{K}^*(x - y).$$

Recalling (3.1), (3.10), (3.16), (3.17), and (3.34) we conclude that Theorem 3.1 is proved. □

3.3 Auxiliary Results for Theorem 3.1

Proof of Lemma 3.1 By using (3.9) and the Christoffel–Darboux formula (3.13–3.14) we get for the r.h.s. of (3.19)

$$\int (\lambda - \mu)^2 K_n^2(\lambda, \mu)d\lambda d\mu = 2(J_{n-1}^{(n)})^2. \tag{3.58}$$

Besides, (2.24) and (3.11–3.12) imply the bound

$$[\psi_l^{(n)}(\lambda)]^2 \le n\rho_n(\lambda) \le n\exp\{-CnV(\lambda)\}, \quad |\lambda| \ge L,\ l = 0, 1, \dots, n-1, \tag{3.59}$$

and then we have by (3.14) that

$$|J_{n-1}^{(n)}| \le C. \tag{3.60}$$

This bound, (1.2) and (3.58) imply (3.19). Similar argument and equation (3.13) yield

$$\int (\lambda - \mu)K_n^2(\lambda, \mu)d\mu = J_{n-1}^{(n)}\psi_{n-1}^{(n)}(\lambda)\psi_n^{(n)}(\lambda). \tag{3.61}$$

Now (3.20) for $\alpha = 1$ follows from this identity and (3.60). The case $\alpha = 2$ in the l.h.s. of (3.20) can be proved similarly and (3.21) follows from (3.20) with $\alpha = 2$. Lemma 3.1 is proved. □

Proof of Lemma 3.2 We start from the simple identity

$$\frac{d\rho_n(\lambda)}{d\lambda} = \left.\frac{d\rho_n(\lambda + t)}{dt}\right|_{t=0}.$$

Changing variables in the integral (2.4) to $\lambda_i - t = \mu_i$, $i = 2, \dots, n$, we rewrite $\rho_n(\lambda + t)$ as

$$\rho_n(\lambda + t) = Q_{n,2}^{-1} \int e^{-nV(\lambda+t)} \prod_{i>j\geq 2}^{n} (\mu_i - \mu_j)^2 \prod_{j=2}^{n} e^{-nV(t+\mu_j)} (\lambda - \mu_j)^2 d\mu_j.$$

Hence, after differentiating with respect to t and setting $t = 0$ in the result we get

$$\begin{aligned} \frac{d\rho_n(\lambda)}{d\lambda} &= -nV'(\lambda)\rho_n(\lambda) - n(n-1)\int V'(\mu) p_{n,2}^{(2)}(\lambda, \mu) d\mu \\ &= -V'(\lambda) K_n(\lambda, \lambda) - \int V'(\mu)(K_n(\lambda, \lambda) K_n(\mu, \mu) - K_n^2(\lambda, \mu)) d\mu, \end{aligned} \tag{3.62}$$

where $p_{n,2}^{(2)}(\lambda, \lambda_2)$ is defined by (2.4) and we used also (3.10) for $l = 2$. Integrating this relation and using (3.11) we obtain

$$\int V'(\mu) K_n(\mu, \mu) d\mu = 0.$$

This, (3.62), and (3.11) yield

$$\rho_n'(\lambda) = \int (V'(\mu) - V'(\lambda)) K_n^2(\lambda, \mu) d\mu. \tag{3.63}$$

We split this integral in two parts corresponding to the intervals $|\mu - \lambda| > d/2$ and $|\mu - \lambda| \leq d/2$, and use (2.29), (2.78), and (3.21) with $\delta = d/2$ for the former integral. In the latter integral we write

$$V'(\mu) - V'(\lambda) = (\mu - \lambda) V''(\lambda) + \frac{(\mu - \lambda)^2}{2} V'''(\xi)$$

for some ξ depending on λ and μ and use Lemma 3.1 and condition (3.5) of Theorem 3.1. Combining the bounds for these two integrals, we obtain (3.23). To obtain (3.22) we use (2.62) for $v = \rho_n$ and (3.23) in the first integral of (2.62) and (2.63) in the second.

To prove (3.24) and (3.25) we introduce the probability density

$$p_n^-(\lambda_1, \dots, \lambda_{n-1}) = \frac{1}{Q_{n,2}^-} \prod_{j=1}^{n-1} e^{-nV(\lambda_j)} \prod_{1\leq j<k\leq n-1} (\lambda_j - \lambda_k)^2. \tag{3.64}$$

The difference of this density from density (2.2) written for $n-1$ variables $\lambda_1, \dots, \lambda_{n-1}$ is that in the former we have the factor n in the exponent while in the latter we would have $n-1$. We have analogously to (3.10) for $l = 1$ and (3.12):

$$\rho_n^-(\lambda) := \frac{n-1}{n} \int p_n^-(\lambda, \lambda_2, \dots, \lambda_{n-1}) d\lambda_2 \dots d\lambda_{n-1} = \frac{1}{n} \sum_{j=0}^{n-2} (\psi_j^{(n)}(\lambda))^2, \tag{3.65}$$

thus

$$(\psi_{n-1}^{(n)}(\lambda))^2 = n(\rho_n(\lambda) - \rho_n^-(\lambda)). \tag{3.66}$$

Furthermore, by using an analog of identity (3.26) for the probability density p_n^-, we obtain the asymptotic relation

$$(f_n^-(z))^2 + \int \frac{V'(\mu)\rho_n^-(\mu)}{\mu - z} d\mu = O(n^{-2}\eta^{-4}) \tag{3.67}$$

for the Stieltjes transform f_n^- of ρ_n^- and $z = \lambda + i\eta,\ \eta > 0$. Denote

$$\Delta_n(z) := n(f_n(z) - f_n^-(z)) = \int \frac{(\psi_{n-1}^{(n)}(\mu))^2}{\mu - z} d\mu, \tag{3.68}$$

subtract (3.67) from (3.26) and multiply the result by n. This yields:

$$\Delta_n(z)(f_n(z) + f_n^-(z)) + \int \frac{V'(\mu)}{\mu - z}(\psi_{n-1}^{(n)}(\mu))^2 d\mu = O(n^{-2}\eta^{-4}).$$

For $z = \lambda + in^{-1/4}$ this relation takes the form

$$\Delta_n(z)(f_n(z) + f_n^-(z) - V'(\lambda)) = \int \frac{V'(\lambda) - V'(\mu)}{\mu - z}(\psi_{n-1}^{(n)}(\mu))^2 d\mu + O(1).$$

Since $\Im f_n^-(z)\Im z > 0, \Im z > 0$, we can write in view of (3.5) and (3.28)

$$0 < \Im\Delta_n(\lambda + in^{-1/4}) \le \left(\frac{1}{\Im f_n(z)}\int \frac{V'(\lambda) - V'(\mu)}{\lambda - \mu}(\psi_{n-1}^{(n)}(\mu))^2 d\mu + O(1)\right) \le C,$$

and then (3.68) for $z = \lambda + in^{-1/4}$ yields:

$$\begin{aligned}\int_{|\mu-\lambda|\le n^{-1/4}} (\psi_{n-1}^{(n)}(\mu))^2 d\mu &\le 2n^{-1/2}\int \frac{(\psi_{n-1}^{(n)}(\mu))^2}{(\mu - \lambda)^2 + n^{-1/2}} d\mu \\ &= 2n^{-1/4}\Im\Delta_n(\lambda + in^{-1/4}) \le Cn^{-1/4}.\end{aligned} \tag{3.69}$$

Hence, we have proved (3.24).

To prove (3.25) for $\psi_{n-1}^{(n)}$ we need two elementary facts. The first is the inequality for a differentiable function $u : [a_1, b_1] \to \mathbb{C}$:

$$\|u\|_\infty^2 \le 2\|u\|_2\|u'\|_2 + (b_1 - a_1)^{-1}\|u\|_2^2, \tag{3.70}$$

where $\|\cdots\|_\infty$ and $\|\cdots\|_2$ are the uniform and the L^2-norm in $[a_1, b_1]$. The inequality (a simple case of the Sobolev inequalities) follows easily from (2.62) with $v = u^2$ and the Schwarz inequality.

The second fact is the identity

$$\int \left(\frac{d}{d\mu}\psi_{n-1}^{(n)}(\mu)\right)^2 d\mu = \int \frac{n^2}{4} V'^2(\mu)(\psi_{n-1}^{(n)}(\mu))^2 d\mu,$$

that follows from (3.7–3.8) and the integration by part, taking into account that $P_k^{(n)}$ is orthogonal to its second derivative, a polynomials of degree $k - 2$. The identity, (2.78), and

(3.9) yield the bound

$$\int \left(\frac{d}{d\mu}\psi_{n-1}^{(n)}(\mu)\right)^2 d\mu \le Cn^2.$$

This, (3.70) for $u = \psi_{n-1}^{(n)}$, $[a_1, b_1] = [\lambda - n^{1/4}, \lambda + n^{1/4}]$, and (3.69) yield (3.25) for $\psi_{n-1}^{(n)}$.

To prove an analogous bounds for $\psi_n^{(n)}$ we repeat the above argument for the probability density (cf. (3.64))

$$p_n^+(\lambda_1, \dots, \lambda_{n+1}) = \frac{1}{Q_{n,2}^+} \prod_{1\le jn+1} e^{-nV(\lambda_j)} \prod_{1\le j<k\le n+1} (\lambda_j - \lambda_k)^2,$$

setting

$$\rho_n^+(\lambda) := \frac{n+1}{n} \int p_n^+(\lambda, \lambda_2, \dots, \lambda_{n+1}) d\lambda_2 \dots d\lambda_{n+1} = \frac{1}{n}\sum_{j=0}^{n} [\psi_j^{(n)}(\mu)]^2,$$

so that $[\psi_n^{(n)}(\lambda)]^2 = n(\rho_n^+(\lambda) - \rho_n(\lambda))$ (cf. (3.66)). Lemma 3.2 is proved. □

Proof of Lemma 3.3 Since A is positive definite there exists a positive definite B such that $A = B^2$. We have then by the Hadamard inequality:

$$\det A = \det B^2 \le \prod_{j=1}^{l} \sum_{k=1}^{l} |B_{jk}|^2.$$

By definition of B the sum in the r.h.s. is $(B^2)_{jj} = A_{jj}$ and we obtain the assertion of lemma. □

Proof of Lemma 3.4 According to (3.28) we have for f_n, defined by (2.82)

$$|\Re f_n(\lambda + i\eta) + V'(\lambda)/2| \le Cn^{-3/8}\log n, \quad \eta = n^{-3/8}. \tag{3.71}$$

On the other hand, using (2.82), integrating by parts the difference $\Re f_n(\lambda + i\eta) - \Re f_n(\lambda + i0)$, written via the r.h.s. of (2.82), and using (3.23) and (3.22), we obtain

$$\begin{aligned}
&\left|\Re f_n(\lambda + i\eta) - \int \frac{\rho_n(\mu)d\mu}{\mu - \lambda}\right| \\
&= \frac{1}{2}\left|\int_{|\mu-\lambda|\le d/2} \log(1 + \eta^2|\mu - \lambda|^{-2})\rho_n'(\mu)d\mu\right| + O(\eta) \\
&\le C\int_{|\mu-\lambda|\le d/2} \log(1 + \eta^2|\mu - \lambda|^{-2})((\psi_{n-1}^{(n)}(\mu))^2 + (\psi_n^{(n)}(\mu))^2)d\mu + O(\eta) \\
&= C(I_1 + I_2 + I_3) + O(\eta),
\end{aligned} \tag{3.72}$$

where d is given in the formulation of Theorem 3.1 and I_1, I_2, and I_3 correspond to the integrals over $|\lambda - \mu| \le n^{-2}$, $n^{-2} \le |\lambda - \mu| \le n^{-1/4}$, and $n^{-1/4} \le |\lambda - \mu| \le d/2$ respectively. Using (3.25) for I_1 and (3.24) for I_2, we get

$$I_1 \le Cn^{-2}\log n \le Cn^{-1}, \qquad I_2 \le Cn^{-1/4}\log n.$$

Besides, for $\eta = n^{-3/8}$ and $|\lambda - \mu| > n^{-1/4}$ we have the inequality $\log(1+\eta^2/|\mu-\lambda|^2) = O(n^{-1/4})$, thus $I_3 = O(n^{-1/4})$. Hence, we obtain from (3.72) that

$$\left| \Re f_n(\lambda + i\eta) - \int \frac{\rho_n(\mu)d\mu}{\mu - \lambda} \right| \le C n^{-1/4} \log n.$$

This inequality and (3.71) prove Lemma 3.4. □

Proof of Lemma 3.5 To simplify notations we denote

$$\lambda_x = \lambda_0 + (x - tx)/n, \quad \lambda_y = \lambda_0 + (y - tx)/n. \tag{3.73}$$

Then, repeating almost literally the derivation of (3.63), we get the formula

$$\frac{d}{dt} K_n(\lambda_x, \lambda_y) = x \int K_n(\lambda_x, \lambda) K_n(\lambda_y, \lambda) \left(\frac{1}{2} V'(\lambda_x) + \frac{1}{2} V'(\lambda_y) - V'(\lambda) \right) d\lambda. \tag{3.74}$$

To estimate the r.h.s. of the formula we split the integral in two parts corresponding to the intervals $|\lambda - \lambda_0| > d/2$ and $|\lambda - \lambda_0| \le d/2$, where $d = \max\{\lambda_0 - a, b - \lambda_0\}$, and for the former integral we use the inequality $2K_n(\lambda, \lambda_x)K_n(\lambda, \lambda_y) \le K_n^2(\lambda, \lambda_x) + K_n^2(\lambda, \lambda_y)$ and then (3.21) with $\delta = d/2$ and (2.78). In the latter integral we write

$$\begin{aligned} &V'(\lambda) - \frac{1}{2} V'(\lambda_x) - \frac{1}{2} V'(\lambda_y) \\ &\quad = \frac{1}{2}(\lambda - \lambda_x) V''(\lambda_x) + \frac{1}{2}(\lambda - \lambda_y) V''(\lambda_y) + O((\lambda - \lambda_x)^2 + (\lambda - \lambda_y)^2) \\ &\quad = \frac{1}{2}(\lambda - \lambda_x) V''(\lambda_x) + \frac{1}{2}(\lambda - \lambda_y) V''(\lambda_y) + O\left((\lambda - \lambda_x)(\lambda - \lambda_y) + \frac{|x-y|^2}{n^2} \right). \end{aligned}$$

The Christoffel–Darboux formula (3.13) yields (cf. (3.61))

$$\int K_n(\lambda_x, \lambda) K_n(\lambda_y, \lambda)(\lambda - \lambda_x) d\lambda = -J_{n-1}^{(n)} \psi_n^{(n)}(\lambda_x) \psi_{n-1}^{(n)}(\lambda_y).$$

Hence

$$\begin{aligned} &\int_{|\lambda - \lambda_0| \le d} K_n(\lambda_x, \lambda) K_n(\lambda_y, \lambda)(\lambda - \lambda_x) d\lambda \\ &\quad = \left(\int - \int_{|\lambda - \lambda_0| \ge d} \right) K_n(\lambda_x, \lambda) K_n(\lambda_y, \lambda)(\lambda - \lambda_{x,y}) d\lambda \\ &\quad = -J_{n-1}^{(n)} \psi_n^{(n)}(\lambda_x) \psi_{n-1}^{(n)}(\lambda_y) - I_d, \end{aligned}$$

where I_d can be estimated by using again (3.21) and an argument similar to that in (3.74). Similar formulas are valid for $(\lambda - \lambda_y)$ in the integrals. Besides, we have by Schwarz inequality,

$$\begin{aligned} &\left| \int K_n(\lambda_x, \lambda) K_n(\lambda_y, \lambda)(\lambda - \lambda_x)(\lambda - \lambda_y) d\lambda \right| \\ &\quad \le \left[\int K_n^2(\lambda_x, \lambda)(\lambda - \lambda_x)^2 d\lambda \int K_n^2(\lambda_y, \lambda)(\lambda - \lambda_y)^2 d\lambda \right]^{1/2}. \end{aligned}$$

Using (3.20) for the r.h.s. of the last inequality and the above estimates for the integrals with $(\lambda-\lambda_x)$ and $(\lambda-\lambda_y)$ we obtain from (3.74)

$$\left|\frac{d}{dt}K_n(\lambda_x,\lambda_y)\right|\le C|x|\bigg((\psi_n^{(n)}(\lambda_x))^2+(\psi_{n-1}^{(n)}(\lambda_x))^2+(\psi_n^{(n)}(\lambda_y))^2 \\ +(\psi_{n-1}^{(n)}(\lambda_y))^2+\frac{|x-y|^2}{n}\bigg). \tag{3.75}$$

The bound, the finite increment formula, and (3.25) imply (3.43). On the other hand, we have

$$\frac{\partial}{\partial x}\mathcal{K}_n(x,y)+\frac{\partial}{\partial y}\mathcal{K}_n(x,y)=-x^{-1}n^{-1}\frac{d}{dt}K_n(\lambda_x,\lambda_y)\bigg|_{t=0}.$$

Combining this with (3.75) and (3.25), we obtain (3.42).

Note, that (3.43) with λ_0+x_1/n instead of λ_0 and $y=x=x_2-x_1$ leads to the bound, valid for any $|x_{1,2}|<nd/2$:

$$|\mathcal{K}_n(x_1,x_1)-\mathcal{K}_n(x_2,x_2)|\le Cn^{-1/8}|x_1-x_2|. \tag{3.76}$$

To prove (3.44) we first show that for any $|x|\le nd/2$ we have the bound

$$\int_{-1}^{1}\frac{\mathcal{K}_n(x,x)\mathcal{K}_n(x+t,x+t)-\mathcal{K}_n^2(x+t,x)}{t^2}dt\le C. \tag{3.77}$$

To this end consider the quantity

$$W=\left\langle\prod_{i=2}^{n}\left|1-\frac{1}{n^2(\lambda_i-\lambda_0)^2}\right|\right\rangle,$$

where the symbol $\langle\cdots\rangle$ denotes the operation $\mathbf{E}\{\delta(\lambda_1-\lambda_0)\cdots\}$ and $\mathbf{E}\{\cdots\}$ is the expectation with respect to the measure (1.1–1.3) for $\beta=2$. By Schwarz inequality W^2 is bounded from above by the product of integrals

$$Z_n^{-1}\int e^{-nV(\lambda_0)}\prod_{2\le j<k\le n}(\lambda_j-\lambda_k)^2\prod_{2\le j\le n}(\lambda_0+\sigma-\lambda_j)^2e^{-nV(\lambda_j)}d\lambda_j$$

for $\sigma=\pm1/n$. Besides, $n(V(\lambda_0)-V(\lambda_0+\sigma))$ is bounded in n because of condition (3.5). Replacing $V(\lambda_0)$ by $V(\lambda_0+\sigma)$ in the above integral and using (2.79) and (3.22) we can write the bound

$$W\le C\cdot\rho_n^{1/2}(\lambda_0+1/n)\rho_n^{1/2}(\lambda_0-1/n)\le C_1. \tag{3.78}$$

On the other hand, W can be represented as

$$W=\left\langle\prod_{i=2}^{n}(\phi_1(\lambda_i)+\phi_2(\lambda_i))\right\rangle=\sum_{k=0}^{n-1}\binom{n-1}{k}\left\langle\prod_{i=2}^{k+1}\phi_1(\lambda_i)\prod_{i=k+2}^{n}\phi_2(\lambda_i)\right\rangle,$$

where

$$\phi_1(\lambda)=\frac{(1-n^2(\lambda-\lambda_0)^2)^2}{n^2(\lambda-\lambda_0)^2}\mathbf{1}_{n|\lambda-\lambda_0|<1},$$

and

$$\phi_2(\lambda) = (1 - n^2(\lambda-\lambda_0)^2)\mathbf{1}_{n|\lambda-\lambda_0|<1} + (1 - n^{-2}|\lambda-\lambda_0|^{-2})\mathbf{1}_{n|\lambda-\lambda_0|>1}.$$

Since $0 \le \phi_2(\lambda) \le 1$ and $\phi_1(\lambda) \ge 0$ we get from the term $k = 1$ of the above representation:

$$W \ge (n-1)\int d\lambda \phi_1(\lambda)\left\langle \delta(\lambda_2 - \lambda)\exp\left\{\sum_{i=3}^{n} \log\phi_2(\lambda_i)\right\}\right\rangle. \tag{3.79}$$

Now the Jensen inequality implies

$$\begin{aligned}
&\left\langle \delta(\lambda_2 - \lambda)\exp\left\{\sum_{i=3}^{n} \log\phi_2(\lambda_i)\right\}\right\rangle \\
&\quad \ge \exp\left\{\left\langle \delta(\lambda_2-\lambda)\sum_{i=3}^{n}\log\phi_2(\lambda_i)[p_{2,2}^{(n)}(\lambda_0,\lambda)]^{-1}\right\rangle\right\} \\
&\quad = \exp\left\{(n-2)\int \log\phi_2(\lambda')p_{3,2}^{(n)}(\lambda_0,\lambda,\lambda')d\lambda'[p_{2,2}^{(n)}(\lambda_0,\lambda)]^{-1}\right\},
\end{aligned} \tag{3.80}$$

where $\langle\delta(\lambda_2-\lambda)\rangle = p_{2,2}^{(n)}(\lambda_0,\lambda)$ and $p_{3,2}^{(n)}(\lambda_0,\lambda,\lambda')$ are the second and the third marginal densities, specified by (2.4) for $\beta = 2$. According to (3.10) for $l = 2, 3$ we have

$$\begin{aligned}
&p_{3,2}^{(n)}(\lambda_0,\lambda,\lambda') \\
&\quad = \frac{n}{n-2}\rho_n(\lambda')p_{2,2}^{(n)}(\lambda_0,\lambda) \\
&\qquad + \frac{2K_n(\lambda_0,\lambda)K_n(\lambda_0,\lambda')K_n(\lambda,\lambda') - K_n(\lambda_0,\lambda_0)K_n^2(\lambda,\lambda') - K_n(\lambda,\lambda)K_n^2(\lambda_0,\lambda')}{n(n-1)(n-2)}.
\end{aligned} \tag{3.81}$$

In view of (3.33) we can write

$$\begin{aligned}
&2K_n(\lambda_0,\lambda)K_n(\lambda_0,\lambda')K_n(\lambda',\lambda) \\
&\quad \le 2K_n^{1/2}(\lambda_0,\lambda_0)K_n^{1/2}(\lambda,\lambda)|K_n(\lambda_0,\lambda')||K_n(\lambda',\lambda)| \\
&\quad \le K_n(\lambda_0,\lambda_0)K_n^2(\lambda',\lambda) + K_n(\lambda,\lambda)K_n^2(\lambda_0,\lambda').
\end{aligned}$$

Thus the second term in the r.h.s. of (3.81) is non-positive and we obtain the bound

$$p_{3,2}^{(n)}(\lambda_0,\lambda,\lambda') \le \frac{n}{n-2}\rho_n(\lambda')p_{2,2}^{(n)}(\lambda_0,\lambda).$$

Hence, taking into account that $\log\phi_2(\lambda) \le 0$ and $\rho_n(\lambda) \le C$, $\lambda \in [a+d, b-d]$ (see (3.22)), restricting the integration in (3.79) by the interval $|\lambda - \lambda_0| \le n^{-1}$, using (3.80–3.81), and recalling the definitions of $\phi_{1,2}$, we have

$$\begin{aligned}
W &\ge (n-1)\int d\lambda\phi_1(\lambda)p_{2,2}^{(n)}(\lambda_0,\lambda)\exp\left\{n\int \rho_n(\lambda')\log\phi_2(\lambda')d\lambda'\right\} \\
&\ge \frac{n-1}{n}\int_{-1}^{1}\frac{(1-t^2)^2}{t^2}p_{2,2}^{(n)}(\lambda_0,\lambda_0+t/n)dt
\end{aligned}$$

$$\times \exp\left\{-C\left(\int_0^1 |\log(1-y^2)|dy + \int_1^\infty \log(1-y^{-2})dy\right) - C\right\}. \quad (3.82)$$

It is easy now to derive (3.77) for $x=0$ from (3.82) and (3.78). Then, replacing λ_0 by $\lambda_0 + x/n$, we obtain the same inequality for any $|x| \le nd/2$.

Now we are ready to prove (3.44). According to (3.39), we have

$$\begin{aligned}\left|\frac{\partial}{\partial x}\mathcal{K}_n(x,y)\right| &= \left|\left(\int_{|x-x'|<1} + \int_{|x-x'|\ge 1}\right)\frac{\mathcal{K}_n(x,x')\mathcal{K}_n(x',y)}{x-x'}dx'\right| + o(1) \\ &\le |I_1(x,y)| + |I_2(x,y)| + o(1). \end{aligned} \quad (3.83)$$

By (3.35) and (3.36) we have

$$|I_2(x,y)| \le \mathcal{K}_n^{1/2}(y,y)\mathcal{K}_n^{1/2}(x,x) \le C.$$

To estimate I_1 denote

$$t_1^* = \inf\{t > 0 : \mathcal{K}_n(x \pm t, x) \le \rho_n(\lambda_0)/2\}, \qquad t^* = \min\{t_1^*, 1\}. \quad (3.84)$$

Then we can write

$$\begin{aligned} I_1(x,y) &= \left(\int_{|x-x'|<t^*} + \int_{t^* \le |x-x'|<1}\right)dx' \\ &\quad \times \frac{\mathcal{K}_n(x,x')\mathcal{K}_n(x',y) - \mathcal{K}_n(x,x)\mathcal{K}_n(x,y)}{x-x'} = I_1' + I_1''. \end{aligned}$$

In view of (3.35–3.36) we have

$$|I_1''| \le C|\log t^*|.$$

On the other hand, using the Schwarz inequality and (3.34), we obtain the bound

$$\begin{aligned} &|\mathcal{K}_n(x,z) - \mathcal{K}_n(x',z)|^2 \\ &\quad = \left|n^{-1}\sum_{k=0}^n \left(\psi_k^{(n)}\left(\lambda_0 + \frac{x}{n}\right) - \psi_k^{(n)}\left(\lambda_0 + \frac{x'}{n}\right)\right)\psi_k^{(n)}\left(\lambda_0 + \frac{x'}{n}\right)\right|^2 \\ &\quad \le (\mathcal{K}_n(x,x) + \mathcal{K}_n(x',x') - 2\mathcal{K}_n(x',x))\mathcal{K}_n(z,z) \\ &\quad = ((\mathcal{K}_n^{1/2}(x,x) - \mathcal{K}_n^{1/2}(x',x'))^2 \\ &\qquad + (\mathcal{K}_n^{1/2}(x,x)\mathcal{K}_n^{1/2}(x',x') - \mathcal{K}_n(x',x))\mathcal{K}_n(z,z). \end{aligned} \quad (3.85)$$

In view of (3.76) and (3.36) the contribution of the first term in the parentheses of the r.h.s. of (3.85) is bounded by $Cn^{-1/4}|x-x'|^2$. Furthermore, write the second term in the parentheses as

$$\mathcal{K}_n^{1/2}(x,x)\mathcal{K}_n^{1/2}(x',x') - \mathcal{K}_n(x',x) = \frac{\mathcal{K}_n(x,x)\mathcal{K}_n(x',x') - \mathcal{K}_n(x',x)}{\mathcal{K}_n^{1/2}(x,x)\mathcal{K}_n^{1/2}(x',x') + \mathcal{K}_n(x',x)}$$

and use the inequality $\mathcal{K}_n(x',x) > \frac{1}{2}\rho_n(\lambda_0) > C$, valid for $|x-x'| \le t^*$. We obtain the bound

$$\begin{aligned}
&|\mathcal{K}_n(x,z)-\mathcal{K}_n(x',z)|^2\\
&\quad\le C(\mathcal{K}_n(x,x)+\mathcal{K}_n(x',x')-2\mathcal{K}_n(x',x))\\
&\quad\le C_1\left(\frac{|x-x'|^2}{n^{1/4}}+\mathcal{K}_n(x,x)\mathcal{K}_n(x',x')-\mathcal{K}_n^2(x',x)\right).
\end{aligned}\tag{3.86}$$

Thus we have from (3.86) with $z=x,y$, (3.36), and the Schwarz inequality

$$\begin{aligned}
|I_1'|&=\left|\int_{|x-x'|\le t^*}\frac{\mathcal{K}_n(x',x)\mathcal{K}_n(x',y)-\mathcal{K}_n(x,x)\mathcal{K}_n(x,y)}{x-x'}dx'\right|\\
&\le C\int_{|x-x'|\le t^*}\frac{|\mathcal{K}_n(x',y)-\mathcal{K}_n(x,y)|}{|x-x'|}dx'\\
&\quad+C\int_{|x-x'|\le t^*}\frac{|\mathcal{K}_n(x,x')-\mathcal{K}_n(x,x)|}{|x-x'|}dx'\le C_1t^*\\
&\quad+C_1\sqrt{t^*}\left(\int_{|x-x'|\le t^*}\frac{|\mathcal{K}_n(x,x)\mathcal{K}_n(x',x')-\mathcal{K}_n^2(x',x)|}{|x-x'|^2}dx'\right)^{1/2}\\
&\le C_2\sqrt{t^*},
\end{aligned}\tag{3.87}$$

where we used (3.77) to estimate the last integral. Now, on the basis of (3.83–3.87) and the finite increment formula, we have that

$$C_1<\rho_n(\lambda_0)/2\le|\mathcal{K}_n(x+t^*,x)-\mathcal{K}_n(x,x)|\le C_2((t^*)^{3/2}+t^*|\log t^*|).$$

We conclude that the inequality $|t^*|\ge d^*$ is valid with some n-independent d^*, hence, repeating derivations of (3.83–3.87) with d^* instead of t^*, we obtain the first inequality of (3.44).

To prove the second inequality in (3.44) we observe first that we have by (3.42):

$$\int_{|x|\le\mathcal{L}}\left|\frac{\partial}{\partial x}\mathcal{K}_n(x,y)\right|^2dx=\int_{|x|\le\mathcal{L}}\left|\frac{\partial}{\partial y}\mathcal{K}_n(x,y)\right|^2dx+o(1),\ |y|\le\mathcal{L}.$$

Then we rewrite an analog of (3.39) for $\frac{\partial}{\partial y}\mathcal{K}_n(x,y)$ as

$$\begin{aligned}
\frac{\partial}{\partial y}\mathcal{K}_n(x,y)&=\left(\int_{|x'-y|\le d^*}+\int_{|x'|\le 2\mathcal{L}}\mathbf{1}_{|x'-y|\ge d^*}\right)\frac{\mathcal{K}_n(x,x')\mathcal{K}_n(x',y)}{y-x'}dx'\\
&\quad+O(\mathcal{L}^{-1})\\
&=I_1(x,y)+I_2(x,y)+O(\mathcal{L}^{-1}).
\end{aligned}$$

Since in I_1 the interval of integration is symmetric with respect to y we can write

$$\begin{aligned}
I_1(x,y)&=\int_{|x'-y|\le d^*}\frac{(\mathcal{K}_n(x,x')-\mathcal{K}_n(x,y))\mathcal{K}_n(x',y)}{y-x'}dx'\\
&\quad+\int_{|x'-y|\le d^*}\frac{\mathcal{K}_n(x,y)(\mathcal{K}_n(x',y)-\mathcal{K}_n(y,y))}{y-x'}dx'.
\end{aligned}$$

Then we have by the Schwarz inequality and (3.36)

$$I_1^2(x,y) \le 2d^* C \int_{|x'-y|\le d^*} \frac{(\mathcal{K}_n(x,x') - \mathcal{K}_n(x,y))^2 dx'}{(y-x')^2} + 2d^* \mathcal{K}_n^2(x,y) \int_{|x'-y|\le d^*} \frac{(\mathcal{K}_n(x',y) - \mathcal{K}_n(y,y))^2}{(y-x')^2} dx'.$$

Now (3.35) and (3.36) lead to the bound

$$\int I_1^2(x,y)dx \le 2d^* C \int_{|x'-y|\le d^*} dx' \frac{\mathcal{K}_n(x',x') + \mathcal{K}_n(y,y) - 2\mathcal{K}_n(x',y)}{(y-x')^2} + 2d^* C \int_{|x'-y|\le d^*} dx' \frac{(\mathcal{K}_n(x',y) - \mathcal{K}_n(y,y))^2}{(y-x')^2}.$$

Using the second inequality of (3.86) for the numerator in the first integral and the first inequality of (3.86) for the numerator in the second integral and then (3.77), we obtain that the integral of $I_1^2(x,y)$ with respect to x is bounded for $|y| \le \mathcal{L}$.

To prove the same I_2 we use (3.35–3.36) to write

$$\begin{aligned}
&\int I_2^2(x,y)dx \\
&\le \int_{|x'|,|x''|\le 2\mathcal{L}} \mathbf{1}_{|x'-y|>d^*} \mathbf{1}_{|x''-y|>d^*} \frac{\mathcal{K}_n(y,x')\mathcal{K}_n(x',x'')\mathcal{K}_n(x'',y)}{(y-x')(y-x'')} dx' dx'' \\
&\le C \int_{|x'|,|x''|\le 2\mathcal{L}} \mathbf{1}_{|x'-y|>d^*} \mathbf{1}_{|x''-y|>d^*} \left(\frac{\mathcal{K}_n^2(y,x')}{(y-x'')^2} + \frac{\mathcal{K}_n^2(y,x'')}{(y-x')^2} \right) dx' dx'' \\
&\le 2C(d^*)^{-1} \mathcal{K}_n(y,y) = O(1).
\end{aligned}$$

The above bounds for integrals of I_1^2 and I_2^2 prove the second inequality in (3.44). Lemma 3.5 is proved. □

Acknowledgements The final version of the paper was written during the authors stay at the H. Poincaré Institute (Paris) in the frameworks of the trimester "Phenomena in High Dimensions". We are grateful to the Organizers of the trimester for hospitality and the CNRS and the Marie Curie Network "Phenomena in High Dimensions" for financial support.

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